

# TRANSFER PRINCIPLES FOR INTEGRABILITY AND BOUNDEDNESS CONDITIONS FOR MOTIVIC EXPONENTIAL FUNCTIONS

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**ABSTRACT.** We prove new transfer principles for motivic exponential functions. Where the first such transfer principle of [R. Cluckers, F. Loeser, *Constructible exponential functions, motivic Fourier transform and transfer principle*, Ann. Math., **171**, 1011-1065 (2010)] treats equalities between integrals, the new transfer principles allow one to change the characteristic of the local field when studying integrability and boundedness conditions. All these transfer principles have applications in the Langlands program. Further we have a theme of results, connecting loci of integrability and loci of boundedness with zero loci and giving new integration results in various settings.

## 1. INTRODUCTION

In this paper we present new transfer principles for motivic exponential functions that allow one to transfer integrability and boundedness conditions from zero characteristic to positive characteristic local fields, and vice versa. These functions were introduced by F. Loeser and the first-named author in [9], where a first transfer principle for motivic exponential functions, dealing with equalities of integrals, was also established. Let us start by giving simple versions of these transfer principles.

In a simplifying approach to motivic exponential functions, one may consider finite collections of (first order) definable functions  $f_i$ ,  $g_{ij}$ , and  $h_i$  in the language of valued fields (for example, polynomials in  $\mathbb{Z}[x_1, \dots, x_n]$ , which could be considered as functions  $\mathcal{O}_K^n \rightarrow K$  on a Cartesian power of the valuation ring of any valued field  $K$ ), with the  $g_{ij}$  non-vanishing. Suppose that each of these functions has the same definable set  $X$  as domain and takes values in the valued field. A drastically simplified motivic exponential function may then be seen as a mere formal expression  $F$  of the form

$$\sum_i |f_i| \cdot E(h_i) \prod_j \text{ord}(g_{ij}).$$

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For any non-archimedean local field  $K$  of large enough residue field characteristic, and for any non-trivial additive character  $\psi : K \rightarrow \mathbb{C}^\times$ , one then obtains an actual, set-theoretic function

$$F_{K,\psi} : X_K \rightarrow \mathbb{C} : x \mapsto \sum_i |f_{iK}(x)| \psi(h_{iK}(x)) \prod_j \text{ord}(g_{ijK}(x)),$$

where the symbols have their obvious meaning as norm and valuation on  $K$ , and where  $X_K$ ,  $f_{iK}$ , and so on, mean interpretation of  $X$ ,  $f_i$ , and so on, in  $K$ . In [9], the non-simplified version of these motivic exponential functions are studied in relation to integration over local fields. To explain the ideas of the transfer principles, the simplified version of motivic exponential functions can serve very well as an intuitive guide.

**Transfer principle of [9].** Suppose that  $K$  is a local field of sufficiently large residue field characteristic. Then, whether

$$F_{K,\psi} \text{ is identically zero for each } \psi$$

holds or not, depends only on the residue field of  $K$ .

This transfer principle of [9] is shown to apply to the Fundamental Lemma of the Langlands program in [6], which yields a general approach to proving that the Fundamental Lemma in characteristic zero follows from the Fundamental Lemma in positive characteristic proved by Ngô [22], as an alternative to the approach by Waldspurger [26]. In this paper we study new transfer principles:

**Transfer principle for integrability.** Suppose that the domain  $X_K$  of  $F_{K,\psi}$  is a subset of  $K^n$  for each local field  $K$ . Then, as soon as the residue field characteristic is sufficiently large, whether

$$F_{K,\psi} \text{ is integrable over } X_K \text{ for each } \psi$$

holds or not, depends only on the residue field of  $K$ .

The same statement is shown for local integrability, and it is shown also in families of motivic functions, which is very important for applications.

**Transfer principle for boundedness.** Suppose that  $K$  is a local field of sufficiently large residue field characteristic. Then, whether

$$F_{K,\psi} \text{ is a bounded function on } X_K \text{ for each } \psi$$

holds or not, depends only on the residue field of  $K$ .

Also the transfer principle for boundedness is shown in a family version, see Theorem 4.4.2.

Before giving a more complete version of our new transfer principles, let us give some context. In his Princeton lecture series in the early seventies, Harish-Chandra introduced what are now called Harish-Chandra characters of irreducible smooth representations of connected reductive  $p$ -adic groups, and he proved in the characteristic zero case that these characters are locally integrable, see [18]. Moreover, still in the characteristic zero case, he

obtained boundedness and local boundedness results for several functions related to Harish-Chandra characters. The positive characteristic analogues of many of these results remained open, although some special cases have meanwhile been established, for example for  $\mathrm{GL}_n$ , see [25] and [20]. These open questions were the original motivation for our new transfer principles, which represents, as far as we can see, the first general strategy to prove this type of results for large characteristics. In [5], we indeed use the new transfer principles to show that Harish-Chandra characters are locally integrable around the unit for admissible representations of unramified reductive groups, and we prove several boundedness results in large positive characteristic.

Note that integrability questions for oscillatory integrals are often very subtle. Let us also stress that these transfer principles are not first order statements in any obvious setting, and so, one has to rely on more geometric and number theoretic techniques than for example for the original Ax-Kochen principle [1], which is more model theoretic.

One of the crucial points in which the non-simplified motivic exponential functions differ from the above simplified presentation, is that they can also invoke formal descriptions of finite field exponential sums, namely over some definable subsets of the residue field. Given a non-Archimedean local field  $K$ , let  $\mathcal{D}_K$  be the collection of additive characters of the field  $K$ , trivial on  $\mathcal{M}_K$  and non-trivial on  $\mathcal{O}_K$ , that project to a fixed character of the residue field under the reduction map, see §4.1 below. Using the notation of [9] which is recalled below in Section 4.2, we can state a version without parameters of the main result of this paper, the new transfer principles.

**Theorem 1.0.1.** *Let  $f$  be a motivic exponential function in  $\mathcal{C}^{\exp}(h[n, 0, 0])$ . Then, there exists  $M > 0$  such that for all non-archimedean local fields  $K$  with residue characteristic larger than  $M$ , the truth of each of the following statements*

- 1)  $f_{K,\psi} : K^n \rightarrow \mathbb{C}$  is integrable for all additive characters  $\psi : K \rightarrow \mathbb{C}^\times$  that lie in  $\mathcal{D}_K$ ,
- 2)  $f_{K,\psi} : K^n \rightarrow \mathbb{C}$  is **locally** integrable for all additive characters  $\psi : K \rightarrow \mathbb{C}^\times$  that lie in  $\mathcal{D}_K$ ,
- 3)  $f_{K,\psi} : K^n \rightarrow \mathbb{C}$  is bounded for all additive characters  $\psi : K \rightarrow \mathbb{C}^\times$  that lie in  $\mathcal{D}_K$ ,
- 4)  $f_{K,\psi} : K^n \rightarrow \mathbb{C}$  is **locally** bounded for all additive characters  $\psi : K \rightarrow \mathbb{C}^\times$  that lie in  $\mathcal{D}_K$ ,

*depends only on the isomorphism class of the residue field of  $K$ .*

Here, we formulated the result for functions on  $K^n$ . However, any variety  $V$  over  $\mathcal{O}_K$  comes with a canonical measure on  $V(K)$  and by general methods, Theorem 1.0.1 implies the same result for functions on  $V(K)$ .

See Theorems 4.4.1 and 4.4.2 and their corollaries for the family versions of Theorem 1.0.1. In fact, we have a theme of results related to integration that we implement in various settings: first in the case of summation over the integers in Section 2, then for  $p$ -adic integration in Section 3, and finally for

motivic integration in Section 4. In this theme we study several kinds of loci, like loci of integrability, loci of boundedness, and loci of identical vanishing (all defined in Section 1.1), and we connect these loci with ordinary zero loci for several classes of functions. Further we give more general integration results than several results of [8] and [9] and we give general interpolation results of given functions by integrable functions. A part of this theme has also been implemented recently in the Euclidean setting in [11] and [10]. For us, the largest difficulties come from the additive characters and their oscillatory behavior, which are absent in the first part about summation over the integers.

In Sections 11.1 and 11.2 of [19], Hrushovski and Kazhdan study distributions in a motivic way, and address integrability issues via a renormalization process by integrating on sets with increasing size. Taking advantage of cancelation on these sets due to oscillation, Hrushovski and Kazhdan can neglect difficulties of Lebesgue integrability. In contrast, we fully address  $L^1$ -integrability, and avoid averaging processes completely.

1.1. We begin by giving the general definitions of loci of integrability, of boundedness, and of identical vanishing. For arbitrary sets  $A \subset X \times T$  and  $x \in X$ , write  $A_x$  for the set of  $t \in T$  with  $(x, t) \in A$ . For  $g : A \subset X \times T \rightarrow B$  a function and for  $x \in X$ , write  $g(x, \cdot)$  for the function  $A_x \rightarrow B$  sending  $t$  to  $g(x, t)$ .

Let  $T$  and  $X$  be arbitrary sets, and let  $f : X \times T \rightarrow \mathbb{C}$  be a function.

**Definition 1.1.1.** Define the *locus of boundedness of  $f$  in  $X$*  as the set

$$\text{Bdd}(f, X) := \{x \in X \mid f(x, \cdot) \text{ is bounded on } T\}.$$

Define the *locus of identical vanishing of  $f$  in  $X$*  as the set

$$\text{Iva}(f, X) := \{x \in X \mid f(x, \cdot) \text{ is identically zero on } T\}.$$

If moreover  $T$  is equipped with a complete measure, we define the *locus of integrability of  $f$  in  $X$*  as the set

$$\text{Int}(f, X) := \{x \in X \mid f(x, \cdot) \text{ is measurable and integrable over } T\}.$$

This definition will mainly be applied to the counting measure on  $\mathbb{Z}$ , to the Haar measure on  $\mathbb{Q}_p$  normalized such that  $\mathbb{Z}_p$  has measure 1, and to Cartesian product measures of these. For sets  $A_1, A_2, X$  and functions  $f_i : A_i \rightarrow X$ , a function  $g : A_1 \rightarrow A_2$  is said to be *over  $X$*  when  $g$  makes a commutative diagram with the maps  $f_i$  to  $X$ . Often, the  $f_i$  will be coordinate projections.

## 2. SUMMABILITY OVER THE INTEGERS

Summation over the integers and the results presented in this section are important for us since they lie behind  $p$ -adic integration: several of the  $p$ -adic results of Section 3, and even some of the motivic results of Section 4, will be reduced to results of this section.

**2.1. Presburger with base  $q$ .** In this section, let  $q > 1$  be a fixed real number.

By a Presburger set, one means a subset of  $\mathbb{Z}^m$  for some  $m \geq 0$  which can be described by a Boolean combination of sets of the following forms

$$\begin{aligned} &\{x \in \mathbb{Z}^m \mid f(x) \geq 0\} \\ &\{x \in \mathbb{Z}^m \mid g(x) \equiv 0 \pmod{n}\}, \end{aligned}$$

where  $f$  and  $g$  are polynomials over  $\mathbb{Z}$  of degree  $\leq 1$ , and  $n > 0$  is an integer. A Presburger function is a function between Presburger sets whose graph is also a Presburger set. A Presburger function is called linear if it is the restriction of an affine map  $\mathbb{Q}^k \rightarrow \mathbb{Q}^\ell$ . We write  $\mathbb{N}$  for the set of non-negative integers  $\{z \in \mathbb{Z} \mid z \geq 0\}$ .

**Definition 2.1.1.** Define the subring  $\mathbb{A}_q \subset \mathbb{R}$  as

$$\mathbb{A}_q = \mathbb{Z} \left[ q, q^{-1}, \left( \frac{1}{1 - q^{-i}} \right)_{i \in \mathbb{N}, 0 < i} \right].$$

For  $S$  a Presburger set, let  $\mathcal{P}_q(S)$  be the  $\mathbb{A}_q$ -algebra of  $\mathbb{A}_q$ -valued functions on  $S$  generated by

- (1) all Presburger functions  $\alpha : S \rightarrow \mathbb{Z}$ ,
- (2) the functions  $q^\beta : S \rightarrow \mathbb{A}_q : s \mapsto q^{\beta(s)}$  for all Presburger functions  $\beta : S \rightarrow \mathbb{Z}$ .

The functions in  $\mathcal{P}_q(S)$  are called Presburger constructible functions on  $S$  (with base  $q$ ). Note that a general function in  $\mathcal{P}_q(S)$  is of the form

$$s \mapsto \sum_{i=1}^N a_i q^{\beta_i(s)} \prod_{j=1}^{M_i} \alpha_{ij}(s),$$

with the  $\alpha_{ij}$  and  $\beta_i$  Presburger functions  $S \rightarrow \mathbb{Z}$ , and the  $a_i$  elements of  $\mathbb{A}_q$ . The constants  $\frac{1}{1 - q^{-i}}$  will be needed in  $\mathbb{A}_q$  to make the framework closed under summation; see Theorem 2.1.6.

By the quantifier elimination results of [24], the image of a Presburger set under a Presburger function is again a Presburger set, as are finite intersections, finite unions, and complements of Presburger sets. The situation for zero loci of Presburger *constructible* functions is much more delicate. For finite unions and finite intersections there is no difficulty, as follows.

**Lemma 2.1.2.** *Consider zero loci*

$$A_i = \{s \in S \mid h_i(s) = 0\}$$

for some  $h_i \in \mathcal{P}_q(S)$ , some Presburger set  $S$ , and  $i = 1, \dots, N$ . Then

$$\bigcap_{i=1}^N A_i, \text{ resp. } \bigcup_{i=1}^N A_i$$

is the zero locus of a function in  $\mathcal{P}_q(S)$ .

*Proof.* One can just take the sum of the squares, resp. the product, of the  $h_i$ .  $\square$

Note that any Presburger subset  $A$  of  $S$  appears as the zero locus of a Presburger constructible function on  $S$  (namely, of the characteristic function of the complement of  $A$ ). However, for  $h \in \mathcal{P}_q(S)$ , the complement of

$$B_h = \{s \in S \mid h(s) = 0\}$$

in  $S$  is not always equal to the zero locus of some function in  $\mathcal{P}_q(S)$ . Still, zero loci of Presburger constructible functions are closely related to loci of integrability, of boundedness, and of identical vanishing. Indeed, the zero loci of Presburger constructible functions are exactly the sets that arise as loci of integrability (against the counting measure) of Presburger constructible functions, and similarly for the loci of boundedness and of identical vanishing.

**Theorem 2.1.3** (Correspondences of loci). *Let  $f$  be in  $\mathcal{P}_q(S \times \mathbb{Z}^m)$  for some Presburger set  $S$  and some  $m \geq 0$ . Then there exist  $h_1, h_2$  and  $h_3$  in  $\mathcal{P}_q(S)$  such that*

$$(2.1.1) \quad \text{Int}(f, S) = \{s \in S \mid h_1(s) = 0\},$$

$$(2.1.2) \quad \text{Bdd}(f, S) = \{s \in S \mid h_2(s) = 0\},$$

and

$$(2.1.3) \quad \text{Iva}(f, S) = \{s \in S \mid h_3(s) = 0\},$$

where integrability in (2.1.1) is with respect to the counting measure on  $\mathbb{Z}^m$ .

*Remark 2.1.4.* Theorem 2.1.3 implies that the classes of sets which can appear as different kinds of loci for Presburger constructible functions are all equal, since for any given function  $h$  in  $\mathcal{P}_q(S)$ , there exists  $f \in \mathcal{P}_q(S \times \mathbb{Z})$  such that

$$\{s \in S \mid h(s) = 0\} = \text{Int}(f, S) = \text{Bdd}(f, S) = \text{Iva}(f, S).$$

Indeed, one can take  $f(s, y) = h(s) \cdot y$ . Hence the name correspondences of loci for Theorem 2.1.3.

One can interpolate Presburger constructible functions by Presburger constructible functions with maximal locus of integrability, as follows.

**Theorem 2.1.5** (Interpolation). *Let  $f$  be in  $\mathcal{P}_q(S \times \mathbb{Z}^m)$  for some Presburger set  $S$  and some  $m \geq 0$ . Then there exists  $g$  in  $\mathcal{P}_q(S \times \mathbb{Z}^m)$  with  $\text{Int}(g, S) = S$  and such that  $f(s, y) = g(s, y)$  whenever  $s$  lies in  $\text{Int}(f, S)$ .*

The following result on stability of  $\mathcal{P}_q$  under summation generalizes Theorem-Definition 4.5.1 of [8] which in turn goes back to Lemma 3.2 of [12]. Theorem-Definition 4.5.1 of [8] is the special case of Theorem 2.1.6 for which  $\text{Int}(f, S) = S$ .

**Theorem 2.1.6** (Integration). *Let  $f$  be in  $\mathcal{P}_q(S \times \mathbb{Z}^m)$  for some Presburger set  $S$  and some  $m \geq 0$ . Then there exists a function  $g \in \mathcal{P}_q(S)$  such that*

$$g(s) = \sum_{y \in \mathbb{Z}^m} f(s, y)$$

*whenever  $s \in \text{Int}(f, S)$ .*

Before proving Theorems 2.1.3, 2.1.5 and 2.1.6, we give some auxiliary results. The first auxiliary lemma is a direct corollary of Wilkie's Theorem of [27] on the  $\mathcal{o}$ -minimality of the real number field with the exponential function.

**Lemma 2.1.7.** *Let  $h : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  be a function of the form*

$$h(x) = \sum_{i=1}^r c_i x^{a_i} b_i^x,$$

*where the  $a_i, b_i, c_i$  are real numbers, the  $c_i$  are nonzero, and  $r \geq 1$ . Suppose that the pairs  $(a_i, b_i)$  are mutually different for different  $i$ . Then the number of zeros of  $f$  is bounded by a constant only depending on  $r$ .*

*Proof.* All functions  $h$  of the above form but with fixed  $r$  are members of a single definable family of functions with discrete zeros in the  $\mathcal{o}$ -minimal structure of the real number field enriched with the exponential function. Now just note that discrete sets which appear as members of a family of sets in an  $\mathcal{o}$ -minimal structure are finite and uniformly bounded in size, cf. [15].  $\square$

**Lemma 2.1.8.** *Let  $h : \mathbb{N}^m \rightarrow \mathbb{R}$  be a function of the form*

$$h(x) = \sum_{i=1}^r c_i q^{b_{i1}x_1 + \dots + b_{im}x_m} \prod_{j=1}^m x_j^{a_{ij}},$$

*where the  $c_i$  are nonzero real numbers, the  $a_{ij}$  and  $b_{ij}$  are integers,  $a_{ij} \geq 0$ ,  $m \geq 1$ , and  $r \geq 1$ . Suppose that the tuples  $(a_{i1}, \dots, a_{im}, b_{i1}, \dots, b_{im})$  are mutually different for different  $i$ . Then  $h$  is not identically zero. Further,  $h$  is summable over  $\mathbb{N}^m$  if and only if  $b_{ij} \leq -1$  for all  $i, j$ . Finally,  $h$  is bounded if and only if at the same time all  $b_{ij}$  are  $\leq 0$  and for each  $i, j$  with  $b_{ij} = 0$  one has  $a_{ij} = 0$ .*

*Proof.* That  $h$  is not identically zero easily follows by induction on  $m$  and by Lemma 2.1.7 for the case  $m = 1$ . If all the  $b_{ij}$  are  $< 0$  then clearly  $h$  is summable. For the other direction, suppose that  $h$  is summable but some  $b_{ij}$  is  $\geq 0$ , say,  $b_{11} \geq 0$ . We may suppose that  $b_{11}$  is maximal among the  $b_{ij}$ . Put  $I = \{i \mid b_{i1} = b_{11}\}$ . We may suppose that  $a_{11}$  is maximal among the  $a_{i1}$  with  $i \in I$ . Put  $J = \{i \in I \mid a_{i1} = a_{11}\}$ . Then the function

$$x \mapsto q^{b_{11}x_1} x_1^{a_{11}} \sum_{i \in J} c_i q^{b_{i2}x_2 + \dots + b_{im}x_m} \prod_{j=2}^m x_j^{a_{ij}}$$

must be identically zero on  $\mathbb{N}^m$  by the summability of  $h$ , which is impossible by the first statement of the lemma. The statement about boundedness is obtained similarly.  $\square$

The following result of [2] will be the basis of the results in this section.

**Theorem 2.1.9** (Parametric Rectilinearization [2]). *Let  $S$  and  $X \subset S \times \mathbb{Z}^m$  be Presburger sets. Then there exists a finite partition of  $X$  into Presburger sets such that for each part  $A$ , there is a set  $B \subset S \times \mathbb{Z}^m$  and a linear Presburger bijection  $\rho : A \rightarrow B$  over  $S$  such that, for each  $s \in S$ , the set  $B_s$  is a set of the form  $\Lambda_s \times \mathbb{N}^\ell$  for a finite subset  $\Lambda_s \subset \mathbb{N}^{m-\ell}$  depending on  $s$  and for an integer  $\ell \geq 0$  only depending on  $A$ .*

Recall that  $B_s$  in Theorem 2.1.9 is the set  $\{z \in \mathbb{Z}^m \mid (s, z) \in B\}$ , and that for  $\rho$  to be over  $S$  means that  $\rho$  makes a commutative diagram with the projections from  $A$  and  $B$  to  $S$ , see Section 1.1.

*Proof of Theorems 2.1.3 and 2.1.5.* We first prove existence of  $h_3$  as in (2.1.3), that is, we first prove the result for  $\text{Iva}(f, S)$ . Since the statement for  $m = 1$  can be applied successively, it is enough to prove the case  $m = 1$ . By Theorem 2.1.9 and since Presburger functions are piecewise linear, there exists a finite partition of  $S \times \mathbb{Z}$  and for each part  $A$  a Presburger bijection  $\rho : A \rightarrow B$  over  $S$  such that either  $B_x = \mathbb{N}$  or  $B_x$  is finite for each  $x \in S$  and such that  $f \circ \rho^{-1}$  is of the form

$$(x, t) \mapsto \sum_{i=1}^r c_i(x) t^{a_i} q^{b_i t}$$

for some integers  $a_i, b_i$  with  $a_i \geq 0$ , and some Presburger constructible functions  $c_i$ , and where the pairs  $(a_i, b_i)$  are mutually different for different  $i$ . By Lemma 2.1.7 there exists a constant  $M \geq 0$  such that, for each fixed value of  $x$ , either the  $c_i(x)$  are all zero for  $i = 1, \dots, r$ , or, there are at most  $M$  zeros of the function  $t \mapsto \sum_{i=1}^r c_i(x) t^{a_i} q^{b_i t}$ . Let us write  $p_A : A \rightarrow S$  for the restriction of the projection  $S \times \mathbb{Z} \rightarrow S$  to  $A$ .

In the case that the cardinality of  $B_x$  is bounded by  $M$  for each  $x$ , one can take  $M$  Presburger functions  $H_1, \dots, H_M$  on  $p_A(A)$  such that the union of the graphs of the  $H_j$  equals  $B$  and then we write for each  $j$

$$Q_{Bj} := \{x \in p_A(A) \mid \sum_{i=1}^r c_i(x) H_j(x)^{a_i} q^{b_i H_j(x)} = 0\}.$$

In the remaining case that the cardinality of  $B_x$  is larger than  $M$  for each  $x$ , write

$$R_{Bi} := \{x \in p_A(A) \mid c_i(x) = 0\}.$$

One can now easily finish for  $h_3$  by Lemma 2.1.2, by taking finite unions and finite intersections of the sets  $Q_{Bj}$ ,  $R_{Bi}$ , and of some Boolean combinations of the Presburger sets  $p_A(A)$ .

Using the existence of  $h_3$  as in (2.1.3) for any given  $f \in \mathcal{P}_q(S \times \mathbb{Z}^m)$ , we now prove jointly Theorem 2.1.5 and the existence of  $h_1$  and  $h_2$ . The



statements clearly allow us to partition  $S \times \mathbb{Z}^m$  into finitely many pieces  $A$  and to treat each one separately (for the existence of  $h_1$  and  $h_2$ , this uses Lemma 2.1.2). We choose a partition such that all Presburger functions involved in  $f$  are Presburger linear, we refine this partition using Theorem 2.1.9, and consider one resulting piece  $A$ . We can replace  $A$  by  $B$  and  $f$  by  $f \circ \rho^{-1}$  with notation from Theorem 2.1.9, so that in the end, we get one Presburger set  $B$  on which we have

$$(2.1.4) \quad f(s, y) = \sum_{i=1}^r c_i(s) y^{a_i} q^{b_i \cdot y}$$

where we use multi-index notation and where  $a_i, b_i \in \mathbb{Z}^m$  with  $a_{ij} \geq 0$ , the  $c_i$  are Presburger constructible functions in  $s \in S$ , the tuples  $(a_i, b_i)$  are mutually different for different  $i$ , and where for each  $s \in S$ , one has  $B_s = \Lambda_s \times \mathbb{N}^\ell$  for a fixed  $\ell \geq 0$  and some finite set  $\Lambda_s \subset \mathbb{N}^{m-\ell}$  depending on  $s$ . In fact, now we are already done by Lemma 2.1.8 and the existence of  $h_3$  as in (2.1.3) for any given  $f$ . Indeed, let  $I$  be  $\{i \mid b_{ij} \geq 0 \text{ for some } j = m - \ell + 1, \dots, m\}$ . Consider the function on  $B$

$$h : (s, y) \mapsto \sum_{i \in I} c_i(s) y^{a_i} q^{b_i \cdot y}$$

for  $s \in S$  and  $y \in B_s$ . Let  $\tilde{h}$  be the extension by zero of  $h$  to a function on  $S \times \mathbb{Z}^m$ . By Lemma 2.1.8, for  $s \in S$ , the family  $\{f(s, y)\}_y$ , where  $y \in B_s$ , is summable if and only if  $\tilde{h}(s, y) = 0$  for all  $y \in \mathbb{Z}^m$ . Since  $\tilde{h}$  is a Presburger constructible function on  $S \times \mathbb{Z}^m$ , we are done for (2.1.1) by the existence of  $h_3$  as in (2.1.3), and also for Theorem 2.1.5 if we define  $g$  piecewise for  $(s, y)$  in  $B$  by

$$g(s, y) = \sum_{i \in \{1, \dots, r\} \setminus I} c_i(s) y^{a_i} q^{b_i \cdot y}.$$

By defining the above set  $I$  slightly differently, one can construct  $h_2$  as in (2.1.2) in a similar way.  $\square$

*Proof of Theorem 2.1.6.* By the interpolation result Theorem 2.1.5, there exists  $g_0$  in  $\mathcal{P}_q(S \times \mathbb{Z}^m)$  with  $\text{Int}(g_0, S) = S$  and such that  $f(s, y) = g_0(s, y)$  whenever  $s$  lies in  $\text{Int}(f, S)$ . Now, by Theorem-Definition 4.5.1 of [8], the function  $g$  which sends  $s \in S$  to  $\sum_{y \in \mathbb{Z}^m} g_0(s, y)$  lies in  $\mathcal{P}_q(S)$ . Clearly  $g$  is as required.  $\square$

Alternatively to invoking Theorem-Definition 4.5.1 of [8] in the above proof of Theorem 2.1.6, one can proceed as follows in that proof to see that  $g$  lies in  $\mathcal{P}_q(S)$ . Use Theorem 2.1.9 to reduce to a sum of the function  $g$  as in (2.1.4), where we may suppose that  $m = 1$ . If  $y = y_1$  runs over  $\mathbb{N}$ , one knows that the  $b_i$  from (2.1.4) are  $< 0$  by the proof of Theorem 2.1.3 and one uses explicit formulas for the summation of geometric power series and their derivatives. When  $\Lambda_s \subset \mathbb{Z}$  is finite, with notation from (2.1.4), one may further assume that  $\Lambda_s$  is of the form  $\{z \in \mathbb{Z} \mid 0 \leq z \leq a(s)\}$ , where  $a$

is a positively valued Presburger function in  $s$  and one uses geometric power series and their derivatives again to sum over  $\Lambda_s$ .

**2.2. Uniformity in the base  $q$ .** We show that the results of Section 2.1 hold uniformly in the base  $q$ . We will use this uniformity in the motivic setting. Write  $\mathbb{R}_{>1}$  for  $\{q \in \mathbb{R} \mid q > 1\}$ .

**Definition 2.2.1.** Define the subring  $\mathbb{A} \subset \mathbb{Q}(\mathbb{L})$  as

$$\mathbb{Z} \left[ \mathbb{L}, \mathbb{L}^{-1}, \left( \frac{1}{1 - \mathbb{L}^{-i}} \right)_{i \in \mathbb{N}, 0 < i} \right],$$

where  $\mathbb{L}$  is a formal symbol. Each  $a \in \mathbb{A}$  is considered as a function

$$(2.2.1) \quad a : \mathbb{R}_{>1} \rightarrow \mathbb{R} : q \mapsto a(q)$$

obtained by setting  $\mathbb{L} = q$ . For  $S$  a Presburger set, let  $\mathcal{P}^u(S)$  be the  $\mathbb{A}$ -algebra of  $\mathbb{R}$ -valued functions on  $S \times \mathbb{R}_{>1}$  generated by

- (1) the functions  $\alpha : S \times \mathbb{R}_{>1} \rightarrow \mathbb{R} : (s, q) \mapsto \alpha(s)$  for all Presburger functions  $\alpha : S \rightarrow \mathbb{Z}$ ,
- (2) the functions  $q^\beta : S \times \mathbb{R}_{>1} \rightarrow \mathbb{R} : (s, q) \mapsto q^{\beta(s)}$  for all Presburger function  $\beta : S \rightarrow \mathbb{Z}$ .

The functions in  $\mathcal{P}^u(S)$  are called Presburger constructible functions on  $S$  with uniform base.

The analogues of Theorems 2.1.3, 2.1.5, and 2.1.6 hold with almost the same proofs.

**Theorem 2.2.2** (Correspondences of loci). *Let  $S$  be a Presburger set and let  $f$  be in  $\mathcal{P}^u(S \times \mathbb{Z}^m)$  for some  $m \geq 0$ . Then there exist  $h_1, h_2$  and  $h_3$  in  $\mathcal{P}^u(S)$  such that*

$$\text{Int}(f, S \times \mathbb{R}_{>1}) = \{(s, q) \in S \times \mathbb{R}_{>1} \mid h_1(s, q) = 0\},$$

$$\text{Bdd}(f, S \times \mathbb{R}_{>1}) = \{(s, q) \in S \times \mathbb{R}_{>1} \mid h_2(s, q) = 0\},$$

and

$$\text{Iva}(f, S \times \mathbb{R}_{>1}) = \{(s, q) \in S \times \mathbb{R}_{>1} \mid h_3(s, q) = 0\}.$$

**Theorem 2.2.3** (Interpolation). *Let  $f$  be in  $\mathcal{P}^u(S \times \mathbb{Z}^m)$  for some Presburger set  $S$  and some  $m \geq 0$ . Then there exists  $g$  in  $\mathcal{P}^u(S \times \mathbb{Z}^m)$  such that  $\text{Int}(g, S \times \mathbb{R}_{>1}) = S \times \mathbb{R}_{>1}$  and such that  $f(s, y, q) = g(s, y, q)$  whenever  $(s, q) \in \text{Int}(f, S \times \mathbb{R}_{>1})$  and  $y \in \mathbb{Z}^m$ .*

**Theorem 2.2.4** (Integration). *Let  $f$  be in  $\mathcal{P}^u(S \times \mathbb{Z}^m)$  for some Presburger set  $S$  and some  $m \geq 0$ . Then there exists a function  $g \in \mathcal{P}^u(S)$  such that*

$$g(s, q) = \sum_{y \in \mathbb{Z}^m} f(s, y, q)$$

whenever  $(s, q) \in \text{Int}(f, S \times \mathbb{R}_{>1})$ .

*Proof of Theorems 2.2.2, 2.2.3, 2.2.4.* Since the Lemmas 2.1.7 and 2.1.8 are completely uniform in  $q$ , the proofs of Section 2.1 go through almost literally the same.  $\square$

### 3. INTEGRABILITY OVER A FIXED $p$ -ADIC FIELD

Let  $K$  be a fixed finite field extension of  $\mathbb{Q}_p$  for a prime number  $p$ . Write  $q_K$  for the number of elements in the residue field  $k_K$  of  $K$ , and  $\mathcal{O}_K$  for the valuation ring of  $K$  with maximal ideal  $\mathcal{M}_K$ . Fix  $\mathcal{L}_K$  to be either the semi-algebraic language on  $K$  with coefficients from  $K$ , that is, Macintyre's language, or the subanalytic language on  $K$  (as in e.g. [16] or [3]). Recall that Macintyre's language is the ring language  $(\cdot, +, -, 0, 1)$  enriched with coefficients from  $K$  and, for each integer  $n > 1$ , a one variable predicate for the set of  $n$ -th powers in  $K^\times$ . The subanalytic language on  $K$  is Macintyre's language enriched with the field inverse  $^{-1}$  on  $K^\times$  extended by  $0^{-1} = 0$ , and for each convergent power series  $f : \mathcal{O}_K^n \rightarrow K$ , a function symbol for the restricted analytic function

$$x \in K^n \mapsto \begin{cases} f(x) & \text{if } x \in \mathcal{O}_K^n, \\ 0 & \text{otherwise.} \end{cases}$$

Note that one has quantifier elimination in  $\mathcal{L}_K$ , by Macintyre's result [21], see also [13], and by [14] for the subanalytic case.

Write  $\varpi_K$  for a fixed uniformizer of  $K$  and write  $|\cdot|$  for the norm on  $K$  with  $|\varpi_K| = q_K^{-1}$ . Put the normalized Haar measure on  $K^n$ , denoted by  $|dx|$  whenever  $x$  are variables running over  $K^n$ , and where the normalization is such that  $\mathcal{O}_K^n$  has measure 1. For each integer  $m > 0$  consider the map  $\overline{\text{ac}}_m : K \rightarrow \mathcal{O}_K/(\varpi_K^m)$  sending nonzero  $x \in K$  to  $\varpi_K^{-\text{ord } x} \cdot x \bmod (\varpi_K^m)$  and sending 0 to 0. We also write  $\overline{\text{ac}}$  for  $\overline{\text{ac}}_1$ . To make the link with the motivic setting easier, we consider three sorted structures for our fixed  $p$ -adic field  $K$ . To this end, we enrich the language  $\mathcal{L}_K$  with the sorts  $\mathbb{Z}$  for the value group, and  $k_K$  for the residue field, together with the valuation map  $\text{ord} : K^\times \rightarrow \mathbb{Z}$  and the angular component map  $\overline{\text{ac}}$ . Let us denote this three-sorted language by  $\mathcal{L}_K^3$ . Let us for each  $m > 0$  identify the map  $\overline{\text{ac}}_m : K \rightarrow \mathcal{O}_K/(\varpi_K^m)$  with a map  $K \rightarrow k_K^m$ , also denoted by  $\overline{\text{ac}}_m$ , by using the bijection  $\mathcal{O}_K/(\varpi_K^m) \rightarrow k_K^m$  which sends  $\sum_{i=0}^{m-1} a_i \varpi_K^i$ , where the  $a_i$  are  $q_K^m$ -th powers in  $\mathcal{O}_K/(\varpi_K^m)$ , to  $(\overline{a_0}, \dots, \overline{a_{m-1}})$ . (To see that this indeed defines a bijection, note that taking the  $q_K^m$ -th power sends  $a + \varpi_K \mathcal{O}_K$  into  $a^{q_K^m} + \varpi_K^m \mathcal{O}_K$  for any unit  $a$  of  $\mathcal{O}_K$  and that  $\text{res}(a^{q_K^m}) = \text{res}(a)$ .) Put on  $K^n \times k_K^m \times \mathbb{Z}^r$  the product topology of the valuation topology on  $K^n$  with the discrete topology on  $k_K^m \times \mathbb{Z}^r$ . In this section, definable will mean  $\mathcal{L}_K^3$ -definable.

**3.1. Constructible functions.** The ring of constructible functions  $\mathcal{C}(X)$  on a definable set  $X$  is the  $\mathbb{A}_{q_K}$ -algebra of real-valued functions on  $X$  generated by functions of the form

- (1)  $f : X \rightarrow \mathbb{Z}$  whenever  $f$  is a definable function,

(2)  $q_K^g : X \rightarrow \mathbb{A}_{q_K} : x \mapsto q_K^{g(x)}$  for definable functions  $g : X \rightarrow \mathbb{Z}$ .

The functions in  $\mathcal{C}(X)$  are called constructible functions on  $X$ .

Now the analogues of Theorems 2.1.3, 2.1.5, and 2.1.6 hold in the  $p$ -adic setting. In fact, the interest of the rings of constructible functions lies in their stability under integration, which we generalize to the following result.

**Theorem 3.1.1** (Integration). *Let  $f$  be in  $\mathcal{C}(X \times K^m)$  for some definable set  $X$  and some  $m \geq 0$ . Then there exists  $g \in \mathcal{C}(X)$  such that*

$$g(x) = \int_{y \in K^m} f(x, y) |dy|$$

whenever  $x \in \text{Int}(f, X)$ .

Under the extra condition that  $\text{Int}(f, X) = X$ , Theorem 3.1.1 was known: in the subanalytic case this is Theorem 4.2 of [3] and the semi-algebraic case has the same proof as in [3], using the semi-algebraic cell decomposition instead of the subanalytic cell decomposition.

**Theorem 3.1.2** (Correspondences of loci). *Let  $f$  be in  $\mathcal{C}(X \times K^m)$  for some definable set  $X$  and some  $m \geq 0$ . Then there exist functions  $h_1, h_2$  and  $h_3$  in  $\mathcal{C}(X)$  such that the zero loci of  $h_i$  equal respectively*

$$\text{Int}(f, X), \quad \text{Bdd}(f, X), \quad \text{and} \quad \text{Iva}(f, X),$$

for  $i = 1, 2$ , resp. 3, and with the normalized Haar measure on  $K^m$ .

Theorem 3.1.2 has the following corollary.

**Corollary 3.1.3.** *Let  $f$  be in  $\mathcal{C}(X \times K^m)$  for some definable set  $X$  and some  $m \geq 0$ . Then there exist functions  $h_1$  and  $h_2$  in  $\mathcal{C}(X)$  such that*

(3.1.1)

$$\{x \in X \mid f(x, \cdot) \text{ is locally integrable on } K^m\} = \{x \in X \mid h_1(x) = 0\}$$

and

$$(3.1.2) \quad \{x \in X \mid f(x, \cdot) \text{ is locally bounded on } K^m\} = \{x \in X \mid h_2(x) = 0\}.$$

*Proof.* Note that local integrability (and similarly for local boundedness) for a function  $r$  on  $K^n$  is equivalent to  $\mathbf{1}_B \cdot r$  being integrable over  $K^n$  (resp. bounded on  $K^n$ ) for each Cartesian product  $B \subset K^n$  of balls in  $K$ , with characteristic function  $\mathbf{1}_B$ . Note that the family of all balls can be (possibly redundantly) realized as the members of a definable family (parameterized by, say, the radius and an element of the ball). Now the Corollary follows from the three statements of Theorem 3.1.2, where the existence of  $h_3$  is used to eliminate the variables that were used to parameterize the balls.  $\square$

**Theorem 3.1.4** (Interpolation). *Let  $f$  be in  $\mathcal{C}(X \times K^m)$  for some definable set  $X$  and some  $m \geq 0$ . Then there exists  $g$  in  $\mathcal{C}(X \times K^m)$  with  $\text{Int}(g, X) = X$  and such that  $f(x, y) = g(x, y)$  whenever  $x$  lies in  $\text{Int}(f, X)$ .*

The above results will be proved using the Cell Decomposition Theorem 3.3.2 below and the analogous results of Section 2.1, but first we state the main  $p$ -adic results in the exponential setting, which will have completely different and more difficult proofs.

**3.2. Constructible exponential functions.** Fix an additive character  $\psi_K : K \rightarrow \mathbb{C}^\times$  which is trivial on  $\mathcal{M}_K$  but nontrivial on  $\mathcal{O}_K$ . (All characters are assumed to be unitary and continuous.) The ring of constructible exponential functions  $\mathcal{C}^{\text{exp}}(X)$  on a definable set  $X$  is the  $\mathbb{A}_{q_K}$ -algebra of complex-valued functions on  $X$  generated by functions of the form

- (1)  $g$  with  $g$  in  $\mathcal{C}(X)$ ;
- (2) functions  $\psi_K(f) : X \rightarrow \mathbb{C} : x \mapsto \psi_K(f(x))$  for any definable function  $f : X \rightarrow K$ .

The functions in  $\mathcal{C}^{\text{exp}}(X)$  are called the constructible exponential functions on  $X$ .

The following analogue of Theorems 2.1.6, 3.1.1 for constructible exponential functions is completely new. A much more restrictive version can be found in [9] (see also [19]).

**Theorem 3.2.1** (Integration). *Let  $f$  be in  $\mathcal{C}^{\text{exp}}(X \times K^m)$  for some definable set  $X$  and some  $m \geq 0$ . Then there exists  $g \in \mathcal{C}^{\text{exp}}(X)$  such that*

$$g(x) = \int_{y \in K^m} f(x, y) |dy|$$

for all  $x \in \text{Int}(f, X)$ .

In [9], Theorem 3.2.1 is proved under an extra condition:  $f$  must be a finite sum of terms of the form  $f_0 \psi_K(f_1)$  with  $f_1 : X \times K^m \rightarrow K$  definable and  $f_0 \in \mathcal{C}(X \times K^m)$  satisfying  $\text{Int}(f_0, X) = X$ .

We also find analogues of Theorems 3.1.2 and 3.1.4 in the exponential setting.

**Theorem 3.2.2** (Correspondences of loci). *Let  $f$  be in  $\mathcal{C}^{\text{exp}}(X \times K^m)$  for some definable set  $X$  and some  $m \geq 0$ . Then there exist functions  $h_1, h_2$  and  $h_3$  in  $\mathcal{C}^{\text{exp}}(X)$  such that*

$$\text{Int}(f, X) = \{x \in X \mid h_1(x) = 0\},$$

$$\text{Bdd}(f, X) = \{x \in X \mid h_2(x) = 0\},$$

and

$$\text{Iva}(f, X) = \{x \in X \mid h_3(x) = 0\}.$$

**Theorem 3.2.3** (Interpolation). *Let  $f$  be in  $\mathcal{C}^{\text{exp}}(X \times K^m)$  for some definable set  $X$ . Then there exists  $g$  in  $\mathcal{C}^{\text{exp}}(X \times K^m)$  with  $\text{Int}(g, X) = X$  and such that  $f(x, y) = g(x, y)$  whenever  $x$  lies in  $\text{Int}(f, X)$ . Moreover, one can write any such  $g$  as a finite sum of terms of the form*

$$f_0 \psi_K(f_1)$$

with  $f_1 : X \times K^m \rightarrow K$  definable and  $f_0 \in \mathcal{C}(X \times K^m)$  satisfying  $\text{Int}(f_0, X) = X$ .

Theorem 3.2.2 implies the following corollary by the same reasoning as for Corollary 3.1.3.

**Corollary 3.2.4.** *Let  $f$  be in  $\mathcal{C}^{\text{exp}}(X \times K^m)$  for some definable set  $X$  and some  $m \geq 0$ . Then there exist functions  $h_1$  and  $h_2$  in  $\mathcal{C}^{\text{exp}}(X)$  such that*

$$(3.2.1) \quad \{x \in X \mid f(x, \cdot) \text{ is locally integrable on } K^m\} = \{x \in X \mid h_1(x) = 0\}$$

and

$$(3.2.2) \quad \{x \in X \mid f(x, \cdot) \text{ is locally bounded on } K^m\} = \{x \in X \mid h_2(x) = 0\}.$$

The following key technical Proposition will allow us to reduce to techniques and results that we have already established in the previous sections. The proposition excludes strange oscillatory behavior of exponential constructible functions.

**Proposition 3.2.5.** *Let  $X$  and  $U \subset X \times K^m$  be definable for some  $m \geq 0$  and let  $f_1, \dots, f_s$  be in  $\mathcal{C}^{\text{exp}}(U)$ . Write  $x$  for variables running over  $X$  and  $y$  for variables running over  $K^m$ . Then there exists an integer  $d \geq 0$ , a definable surjection  $\varphi : U \rightarrow V \subset X \times \mathbb{Z}^t$  over  $X$  for some  $t \geq 0$ , definable functions  $h_{\ell i} : U \rightarrow K$ , and functions  $G_{\ell i}$  in  $\mathcal{C}^{\text{exp}}(V)$  for  $\ell = 1, \dots, s$ , such that the following conditions hold.*

1) for each  $\ell$ , one has

$$f_{\ell}(x, y) = \sum_{i=1}^{N_{\ell}} G_{\ell i}(\varphi(x, y)) \psi_K(h_{\ell i}(x, y)),$$

2) if one sets, for  $(x, r) \in V$ ,

$$U_{x,r} := \{y \in U_x \mid \varphi(x, y) = (x, r)\}$$

and

$$W_{x,r} := \{y \in U_{x,r} \mid \sup_{\ell, i} |G_{\ell i}(x, r)|_{\mathbb{C}} \leq \sup_{\ell} |f_{\ell}(x, y)|_{\mathbb{C}}\},$$

where  $|\cdot|_{\mathbb{C}}$  is the complex modulus, then

$$\text{Vol}(U_{x,r}) \leq q_K^d \cdot \text{Vol}(W_{x,r}) < +\infty,$$

where the volume  $\text{Vol}$  is taken with respect to the Haar measure on  $K^m$ .

Roughly, the proposition for  $s = 1$  says that, if  $|f_1|_{\mathbb{C}}$  is small, then  $f_1$  is the sum of small terms of a very specific form. Indeed, for the terms to be small, it suffices to know that they are small on each set  $W_{x,r}$  since they are constant on each set  $U_{r,x}$ . In addition, if these terms cannot be made small, then  $f$  itself has to be large on a relatively large set.

For the proposition to make sense, the sets  $U_{x,r}$  and  $W_{x,r}$  have to be measurable, but this follows from the facts that each definable set is measurable, functions in  $\mathcal{C}^{\text{exp}}(Z)$  are measurable for any definable  $Z$ , and, that measurable functions are closed under taking the complex modulus and the supremum.

**3.3. Preliminaries for the  $p$ -adic proofs.** We give a notion of  $p$ -adic cells which is adapted to the three sorts in  $\mathcal{L}_K^3$  and which fits better with the motivic approach below, see e.g. the usage of  $\xi$  in the next definition.

**Definition 3.3.1** ( $p$ -adic cells). Let  $Y$  be a definable set. A 1-cell  $A \subset Y \times K$  over  $Y$  is a (nonempty) set of the form

$$(3.3.1) \quad A = \{(y, t) \in Y' \times K \mid \alpha(y) \sqsubset_1 \text{ord}(t - c(y)) \sqsubset_2 \beta(y), \\ \text{ord}(t - c(y)) \in a + n\mathbb{Z}, \overline{\text{ac}}_m(t - c(y)) = \xi(y)\},$$

with  $Y'$  a definable subset of  $Y$ , integers  $a \geq 0$ ,  $n > 0$ ,  $m > 0$ ,  $\alpha, \beta: Y' \rightarrow \mathbb{Z}$  and  $\xi: Y' \rightarrow (\mathcal{O}_K/(\varpi_K^m))^\times$  definable,  $c: Y' \rightarrow K$  definable, and  $\sqsubset_i$  either  $<$  or no condition, and such that  $A$  projects surjectively onto  $Y'$ . We call  $c$  the center,  $\xi$  the angular component,  $a + n\mathbb{Z}$  the coset,  $\alpha$  and  $\beta$  the boundaries, and  $Y'$  the base of  $A$ . A 0-cell  $A \subset Y \times K$  over  $Y$  is a (nonempty) set of the form

$$(3.3.2) \quad A = \{(y, t) \in Y' \times K \mid t = c(y)\},$$

with  $Y'$  a definable subset of  $Y$ , and  $c: Y' \rightarrow K$  definable. In both cases we call  $A$  a cell over  $Y$  with center  $c$ .

Also our formulation of the cell decomposition result is somehow more relaxed than usual, due to having three sorts. For a slightly stronger cell decomposition result than Theorem 3.3.2, see e.g. [4], Theorem 3.3.

**Theorem 3.3.2** ( $p$ -adic Cell Decomposition). *Let  $X \subset Y \times K$  and  $f_j: X \rightarrow \mathbb{Z}$  be definable for some definable set  $Y$  and  $j = 1, \dots, r$ . Then there exists a finite partition of  $X$  into cells  $A_i$  (over  $Y$ ) with center  $c_i$  such that for each occurring 1-cell  $A_i$  with coset  $a_i + n_i\mathbb{Z}$  and base  $Y'_i$  one has*

$$f_j(y, t) = h_{ij}(y) + a_{ij} \frac{\text{ord}(t - c_i(y)) - a_i}{n_i}, \quad \text{for each } (y, t) \in A_i,$$

with integers  $a_{ij}$  and  $h_{ij}: Y'_i \rightarrow \mathbb{Z}$  definable functions for  $j = 1, \dots, r$ .

For  $X \subset K$  open, a function  $f: X \rightarrow K$  is called  $C^1$  if  $f$  is differentiable at each point of  $X$  and the derivative  $f': X \rightarrow K$  of  $f$  is continuous. (This notion of  $C^1$ , although more naive than the ones in e.g. [17], suffices for our purposes.) A ball in  $K$  is by definition a set of the form  $\{t \in K \mid \text{ord}(t - a) \geq z\}$  for some  $a \in K$  and some  $z \in \mathbb{Z}$ .

**Definition 3.3.3** (Jacobian property). Let  $f: B_1 \rightarrow B_2$  be a function with  $B_1, B_2 \subset K$ . Say that  $f$  has the Jacobian property if the following conditions a) up to d) hold:

- a)  $f$  is a bijection from  $B_1$  onto  $B_2$  and  $B_1$  and  $B_2$  are balls in  $K$ ;
- b)  $f$  is  $C^1$  on  $B_1$  with nonvanishing derivative  $f'$ ;
- c)  $|f'|$  is constant on  $B_1$ ;
- d) for all  $x, y \in B_1$  one has

$$|(x - y) \cdot f'| = |f(x) - f(y)|.$$

**Definition 3.3.4** (1-Jacobian property). Let  $f: B_1 \rightarrow B_2$  be a function with  $B_1, B_2 \subset K$ . Say that  $f$  has the 1-Jacobian property if  $f$  has the Jacobian property and moreover e) and f) hold:

- e)  $\overline{\text{ac}}(f')$  is constant on  $B_1$ ;
- f) for all  $x, y \in B_1$  one has

$$\overline{\text{ac}}(f') \cdot \overline{\text{ac}}(x - y) = \overline{\text{ac}}(f(x) - f(y)).$$

The following two results will be important for the proofs in the exponential setting.

**Proposition 3.3.5** ([7], Section 6). *Let  $Y$  and  $X \subset Y \times K$  be definable sets and let  $F: X \rightarrow K$  be a definable function. Then there exists a finite partition of  $X$  into cells  $A_i$  over  $Y$  such that for each  $i$  and each  $y \in Y$ , the restriction of  $F(y, \cdot)$  to  $A_{iy} := \{t \in K \mid (y, t) \in A_i\}$  is either constant or injective, and such that in the latter case, for each ball  $B \subset K$  such that  $\{y\} \times B$  is contained in  $A_i$ , there is a ball  $B^* \subset K$  such that  $F(y, B) = B^*$  and such that the map*

$$F_B: B \rightarrow B^*: t \mapsto F(y, t)$$

*has the 1-Jacobian property.*

**Lemma 3.3.6.** *Let  $A \subset Y \times K$  and  $h: A \rightarrow K$  be definable for some definable set  $Y$ . Suppose that for each  $y \in Y$ , and for each maximal ball  $B \subset A_y$ ,  $h(y, \cdot)$  is constant modulo  $(\varpi_K)$  on  $B$ , where maximality is for the inclusion. Then there exists a finite partition of  $A$  into definable sets  $A_j$  and definable functions  $h_j: Y \rightarrow K$  such that*

$$|h(y, t) - h_j(y)| \leq 1$$

*holds for each  $(y, t) \in A_j$  and for each  $j$ .*

*Remark 3.3.7.* We could even obtain  $|h(y, t) - h_j(y)| < 1$ . However, we would like to have proofs which work analogously in the motivic setting, and there, the stronger inequality can not be obtained.

*Proof of Lemma 3.3.6.* The proof is similar to the proof of Theorem 2.2.2 of [9] and goes as follows. Up to a finite partition of  $A$ , we may suppose that for each  $y \in Y$ , the function  $h(y, \cdot)$  is injective (see e.g. Corollary 3.7 of [4]). Similarly we may suppose that  $h$  is as  $F$  in the conclusion of Proposition 3.3.5 already on the whole of  $A$ . We may moreover for each two balls  $\{y\} \times B_1$  and  $\{y\} \times B_2$  contained in  $A$  assume that the images  $h(\{y\} \times B_1)$  and  $h(\{y\} \times B_2)$  are balls with different radius, for example by invoking Theorem 3.3.2. Now consider the graph of  $h$  in  $A \times K$ , and its image  $W \subset Y \times K$  under



the coordinate projection sending  $(y, t, h(y, t))$  to  $(y, h(y, t))$ . Take a cell decomposition as in Theorem 3.3.2 of  $W$  into cells  $C_j$ . Up to partitioning  $A$  into finitely many parts we may suppose  $W = C_j = C$  for a single  $j$ , where we do not require to respect the previous assumptions on  $A$ . Write  $c$  for the center of  $C$ . Suppose that the angular component of  $C$  takes values in  $\mathcal{O}_K/(\varpi_K^m)$  for some  $m > 0$  in the case that  $C$  is a 1-cell. If  $C$  is a 0-cell then put  $m = 1$ . Now it follows for all  $(y, t) \in A$  that

$$|h(y, t) - c(y)| \leq q_K^{m-1}.$$

If  $m = 1$  we are done by taking  $h_j = c$  on  $A = A_j$  for a single  $j$ . If  $m > 1$  we can finish by the definability of Skolem functions and by further partitioning.  $\square$

**3.4. The  $p$ -adic proofs for constructible functions.** For  $p$ -adic constructible functions (thus without additive character), the proofs reduce to the Presburger cases of Section 2.1 with  $q = q_K$ , via  $p$ -adic cell decomposition and the following definitions and results.

**Definition 3.4.1.** If  $f_j: X \subset K^{m+1} \rightarrow \mathbb{Z}$  and the  $A_i$  are as in Theorem 3.3.2, then call  $f_j$  *prepared* on the cells  $A_i$ . We call  $A_i$  a *full cell* and we call  $f_j$  *fully prepared* on the  $A_i$  if the base of  $A_i$  is itself a cell on which the  $h_{ij}(x)$ —with notation from Theorem 3.3.2 for 1-cells and with  $h_{ij} = f_j$  in the case of 0-cells—and the boundaries of  $A_i$  are prepared, and so on  $m$  times. It is also clear what we mean by a full cell  $A \subset Y \times K^{m+1}$  over some definable set  $Y$ , in analogy to the notion of cells over  $Y$  of Definition 3.3.1. By the centers of a full cell, we mean a tuple of centers, consisting of the center of the cell  $A$  over  $Y$ , the center of the base  $A'$  of  $A$ , the center of the base of  $A'$  and so on.

**Definition 3.4.2.** Let  $A \subset K^m$  be a full cell with center  $c_m$  in the last coordinate, up to center  $c_1 \in K$  for the first coordinate. The skeleton of  $A$  is then the subset  $S(A)$  of  $(\mathbb{Z} \cup \{+\infty\})^m$  which is the image of  $A$  under the map

$$x \in A \mapsto (\text{ord}(x_1 - c_1), \text{ord}(x_2 - c_2(x_1)), \dots, \text{ord}(x_m - c_m(x_1, \dots, x_{m-1}))),$$

where we have extended  $\text{ord}$  to a map  $\text{ord}: K \rightarrow \mathbb{Z} \cup \{+\infty\}$ . Write  $s_A$  for the natural map  $A \rightarrow S(A)$  which we call the skeleton map. Likewise, if  $A \subset Y \times K^\ell$  is a full cell over  $Y$ , it is clear what we mean by the skeleton  $S_Y(A)$  over  $Y$  and the skeleton map  $s_{A/Y}$  of  $A$  over  $Y$ .

Call a definable function  $f: X \subset Y \times \mathbb{Z}^m \rightarrow Y \times \mathbb{Z}^\ell$  linear over  $Y$  if there is a definable function  $a: Y \rightarrow \mathbb{Z}^\ell$  and an affine map  $g: \mathbb{Q}^m \rightarrow \mathbb{Q}^\ell$  such that  $f(y, z) = (y, g(z) + a(y))$  for all  $(y, z) \in X$ .

**Proposition 3.4.3** (Parametric Rectilinearization for  $\mathcal{L}_K^3$ ). *Let  $Y$  and  $X \subset Y \times \mathbb{Z}^m$  be definable sets. Then there exists a finite partition of  $X$  into definable sets such that for each part  $A$ , there is a set  $B \subset Y \times \mathbb{Z}^m$  and a definable bijection  $\rho: A \rightarrow B$  which is linear over  $Y$  such that, for each*

$y \in Y$ , the set  $B_y$  is a set of the form  $\Lambda_y \times \mathbb{N}^\ell$  for a finite subset  $\Lambda_y \subset \mathbb{N}^{m-\ell}$  depending on  $y$  and for an integer  $\ell \geq 0$  only depending on  $A$ .

*Proof.* This follows from Theorem 2.1.9 as follows. Suppose that  $Y \subset \mathbb{Z}^r \times K^\ell \times k_K^n$ , and let  $X' \subset K^{r+\ell+n+m}$  be the inverse image of  $X$  under the map sending  $(z, u, v, w)$  to the tuple  $(\text{ord}(z_i), u, \overline{\text{ac}}(v_j), \text{ord}(w_k))$ . Partition  $X'$  into full cells  $A_i$ , take skeletons and apply Theorem 2.1.9 to the skeletons. Translating the data back to the level of  $X$  finishes the proof. Alternatively, one can just note that the proof of [2] goes through in this setting with three sorts.  $\square$

*Proof of Theorems 3.1.2 and 3.1.4.* By using an inductive application of the Cell Decomposition Theorem 3.3.2, partition  $X \times K^m$  into finitely many full cells  $A_i$  over  $X$  such that for each  $i$ , the restriction  $f|_{A_i}$  factorizes through the skeleton map  $s_{A_i/X}$  of  $A_i$  over  $X$ . Let us identify each skeleton with a definable set, for example by replacing  $\{+\infty\}$  by a disjoint copy of  $\{0\}$ . Let us write  $f_i$  for the map from the skeleton  $S(A_i)$  of  $A_i$  to  $\mathbb{A}_{q_K}$  induced by  $f|_{A_i}$ . Then  $f_i$  lies in  $\mathcal{P}_{q_K}(S(A_i))$  for each  $i$ . The function  $M_z$  sending  $z \in S_{/X}(A_i)$  to the volume of the fiber  $(s_{A_i/X})^{-1}(z)$ , taken inside  $K^m$ , lies in  $\mathcal{C}(S_{/X}(A_i))$ . Indeed, the fibers of  $s_{A_i/X}$  are of a very simple nature by the definition of full cells. Hence, also  $z \mapsto f_i(z) \cdot M_z$  lies in  $\mathcal{C}(S_{/X}(A_i))$ . Now existence of  $h_2$  and  $h_3$  is proved as in the Presburger case in Theorem 2.1.3 for the  $f_i$ , and the existence of  $h_1$  and the construction of  $g$  as in Theorem 3.1.4 are also proved as in the Presburger cases for the maps  $z \mapsto f_i(z) \cdot M_z$ .  $\square$

*Proof of Theorem 3.1.1.* Follows from Theorem 3.1.4 and [3, Theorem 4.2], in the same way as Theorem 2.1.6 follows from Theorem 2.1.5 and [8, Theorem-Definition 4.5.1].  $\square$

In fact, the proof of Theorems 3.1.2 and 3.1.4 yields the following slightly more general variant.

**Corollary 3.4.4.** *Let  $f$  be in  $\mathcal{H}(X) \otimes_{\mathcal{C}(X)} \mathcal{C}(X \times K^m)$  for some definable set  $X$ , some  $m \geq 0$ , and some inclusion  $\mathcal{C}(X) \subset \mathcal{H}(X)$  of algebras of complex valued functions on  $X$ . Then there exist functions  $h_1, h_2, h_3$  in  $\mathcal{H}(X)$  and  $g$  in  $\mathcal{H}(X) \otimes_{\mathcal{C}(X)} \mathcal{C}(X \times K^m)$  such that the zero loci of the  $h_i$  equal respectively*

$$\text{Int}(f, X), \quad \text{Bdd}(f, X), \quad \text{and} \quad \text{Iva}(f, X),$$

*and such that  $\text{Int}(g, X) = X$  and  $f(x, y) = g(x, y)$  whenever  $x$  lies in  $\text{Int}(f, X)$ . Moreover, any such  $g$  can be written as a finite sum of terms of the form*

$$h_0 \cdot f_0$$

*with  $h_0 \in \mathcal{H}(X)$  and  $f_0 \in \mathcal{C}(X \times K^m)$  satisfying  $\text{Int}(f_0, X) = X$ .*

**3.5. The  $p$ -adic proofs for constructible exponential functions.** Consider a finite field  $\mathbb{F}_q$  with a nontrivial additive character  $\psi$ . The following lemma and its corollary are classical exercises.

**Lemma 3.5.1.** *For any function  $f : \mathbb{F}_q \rightarrow \mathbb{C}$  one has*

$$\frac{1}{q} \|\hat{f}\|_{\text{sup}} \leq \|f\|_{\text{sup}} \leq \|\hat{f}\|_{\text{sup}}$$

where  $\|\cdot\|_{\text{sup}}$  is the supremum norm and  $\hat{f}$  the Fourier transform of  $f$ ,

$$\hat{f}(y) = \sum_{x \in \mathbb{F}_q} f(x) \psi(-xy).$$

**Corollary 3.5.2.** *Consider a function*

$$f : \mathbb{F}_q \rightarrow \mathbb{C} : y \mapsto \sum_{j=1}^s c_j \psi(b_j y)$$

for some complex numbers  $c_j$  and some mutually different  $b_j \in \mathbb{F}_q$ . Then there exists  $y_0 \in \mathbb{F}_q$  with

$$\sup_{j=1}^s |c_j| \leq f(y_0).$$

*Proof of Proposition 3.2.5 for  $m = 1$ .* The statement which we have to prove clearly allows us to work piecewise: if we have a finite partition of  $U$  into definable parts  $A$ , then it suffices to prove the proposition for  $f_\ell$  restricted to each part  $A$ . To make the induction on  $N$  work more easily, we prove something slightly stronger than Proposition 3.2.5 purely for the case  $m = 1$ . That is, we prove that in addition to the conclusions 1) and 2) of the Proposition, we can require that also the following condition 3) holds for each  $x \in X$ .

- 3) Each of the parts  $U_{x,r}$  is either a singleton, or, equal to a maximal ball contained in  $U_x$ , where maximality is for the inclusion.

By Theorem 3.3.2, up to partitioning  $U$  into cells  $A$ , we may suppose that  $U$  is a cell. If  $U$  is a 0-cell over  $X$ , there is nothing to prove. If  $U$  is a 1-cell over  $X$ , up to a finite partition given by Theorem 3.3.2 (applied to all the  $\mathbb{Z}$ -valued definable functions that appear in the build up of the  $f_\ell$ ), we may suppose that there is a definable surjection  $\varphi : U \rightarrow V \subset X \times \mathbb{Z}^t$ , definable functions  $h_{\ell i} : U \rightarrow K$  and functions  $G_{\ell i}$  in  $\mathcal{C}^{\text{exp}}(V)$  such that for each  $\ell$  one has

$$(3.5.1) \quad f_\ell(x, y) = \sum_{i=1}^{N_\ell} G_{\ell i}(\varphi(x, y)) \psi_K(h_{\ell i}(x, y)),$$

and that, for each  $x$ , the collection of sets  $U_{x,r}$  equals the collection of maximal balls contained in  $U_x$ . We now only have to show that one can choose  $\varphi$  such that moreover 2) holds. We will proceed by induction on  $N := \sum_{\ell=1}^s (N_\ell - 1)$ . If  $N = 0$ , then all  $N_\ell = 1$ , and one is done.

For general  $N > 0$  we start by pulling out a factor  $\psi_K(h_{\ell 1}(x, y))$  out of (3.5.1), i.e., we may assume that  $h_{\ell 1}(x, y) = 0$  for all  $\ell$  and all  $(x, y) \in U$ . By Proposition 3.3.5, and by Theorem 3.3.2, we may moreover suppose that for each  $(x, r) \in V$ , each  $\ell$ , and each  $i$  either  $h_{\ell i}(x, \cdot)$  is constant on  $U_{x,r}$ ,

or  $h_{\ell i}(x, \cdot)$  restricted to  $U_{x,r}$  has the 1-Jacobian property. Hence, for each  $(x, r) \in V$  there exist constants  $b_{x,r,\ell,i} \in K$  such that, for all  $y_1, y_2 \in U_{x,r}$  and all  $\ell, i$ ,

$$(3.5.2) \quad \text{ord}(h_{\ell i}(x, y_1) - h_{\ell i}(x, y_2)) = \text{ord}(b_{x,r,\ell,i} \cdot (y_1 - y_2)),$$

$$(3.5.3) \quad \overline{\text{ac}}(h_{\ell i}(x, y_1) - h_{\ell i}(x, y_2)) = \overline{\text{ac}}(b_{x,r,\ell,i} \cdot (y_1 - y_2)),$$

where  $b_{x,r,\ell,1} = 0$  by a previous assumption, and where we write  $\text{ord} : K \rightarrow \mathbb{Z} \cup \{+\infty\}$ . If for all  $\ell, i, x, r$ , the function  $h_{\ell i}(x, \cdot)$  is constant modulo  $(\varpi_K)$  on  $U_{x,r}$ , then, up to a further finite partition of  $U$ , Lemma 3.3.6 applied to each of the  $h_{\ell i}$  brings us back in the case that  $N = 0$ . We may thus in particular assume that for each  $(x, r)$  in  $V$ , there exist  $\ell, i$  with  $b_{x,r,\ell,i} \neq 0$ . Choose  $\gamma_{x,r} \in K$  with

$$|\gamma_{x,r}| \cdot \max_{\ell,i} |b_{x,r,\ell,i}| = 1.$$

For each  $x$  and  $r$ , partition  $\{1, \dots, N_\ell\}$  into non-empty subsets  $S_{\ell j}(x, r)$ ,  $j \geq 1$ , with the property that  $i_1, i_2$  lie in the same part  $S_{\ell j}(x, r)$  for some  $j$  if and only if

$$(3.5.4) \quad \text{res}(\gamma_{x,r} b_{x,r,\ell,i_1}) = \text{res}(\gamma_{x,r} b_{x,r,\ell,i_2}),$$

where  $\text{res} : \mathcal{O}_K \rightarrow k_K$  is the natural projection. By cutting  $U$  into finitely many pieces again, we may suppose that the sets  $S_{\ell j} := S_{\ell j}(x, r)$  do not depend on  $(x, r)$ . Since  $b_{x,r,\ell,1} = 0$ , at least for one  $\ell$  there are at least two different sets  $S_{\ell,j}, S_{\ell,j'}$ . Define for each  $\ell, j$  and for  $(x, y) \in U$

$$f_{\ell j}(x, y) := \sum_{i \in S_{\ell j}} G_{\ell i}(\varphi(x, y)) \psi_K(h_{\ell i}(x, y))$$

and consider these functions  $(f_{\ell j})_{\ell,j}$  as a single family. There is no harm in replacing  $U$  by the graph of  $\varphi$  and then proving the Proposition for this graph. Indeed, one can always get back to the original  $U$  by composing with the previous map  $\varphi$ , and by noting that the  $K$ -valued definable functions  $h_{\ell i}$  can only depend piecewise trivially on  $\mathbb{Z}$ -variables. So, we may suppose that  $\varphi$  is in fact just a coordinate projection, and it will play no role anymore in the later part of the proof. The total number of summands of the family  $(f_{\ell j})_{\ell,j}$  is the same as for the functions  $f_\ell$ , but there are more functions  $f_{\ell j}$  than  $f_j$ , so we can apply induction on  $N$ , with the extra condition 3) as part of the desired properties, to this family  $(f_{\ell j})_{\ell,j}$ . Thus we find an integer  $d \geq 0$ , a definable surjection  $\varphi : U \rightarrow V$  over  $X$ , definable functions  $h_{\ell j i} : U \rightarrow K$ , and functions  $G_{\ell j i}$  with the required properties, including condition 3). Let us also write  $U_{x,r}$  for the sets defined by  $\varphi$  as in condition 2). By cutting  $U$  into pieces as before, we may assume that, for each  $x$  and  $r$ , not all  $h_{\ell i}(x, \cdot)$  are constant modulo  $(\varpi_K)$  on  $U_{x,r}$ , since, as before, this would bring us back to the  $N = 0$  case for our original family  $(f_\ell)_\ell$  via Lemma 3.3.6.

We will now show that the subset  $M_{x,r}$  of  $U_{x,r}$  consisting of those  $y$  satisfying both inequalities

$$(3.5.5) \quad \sup_{\ell,j,i} |G_{\ell ji}(x,r)|_{\mathbb{C}} \leq \sup_{\ell,j} |f_{\ell j}(x,y)|_{\mathbb{C}} \leq \sup_{\ell} |f_{\ell}(x,y)|_{\mathbb{C}}$$

has big volume in the sense that

$$(3.5.6) \quad \text{Vol}(U_{x,r}) \leq q_K^{d+1} \text{Vol}(M_{x,r}).$$

If this is proved, then we are done for our original family  $(f_{\ell})_{\ell}$  by replacing  $d$  with  $d+1$  while keeping the data of the  $\varphi$ ,  $G_{\ell ji}$ , and  $h_{\ell ji}$ .

To finish the proof, we fix  $x$  and  $r$  and it remains to show that  $M_{x,r}$  as given by (3.5.5) has property (3.5.6). Consider the partition of the ball  $U_{x,r}$  into balls  $B_{\xi}$  of the form  $\xi + \gamma_{x,r} \mathcal{O}_K$ . (The ball  $U_{x,r}$  is indeed a union of such balls  $B_{\xi}$  by our choice of  $\gamma_{x,r}$  since there exists a  $h_{\ell i}(x, \cdot)$  which is non-constant modulo  $(\varpi_K)$  on  $U_{x,r}$ .) Firstly we will show that  $|f_{\ell j}(x, \cdot)|_{\mathbb{C}}$  is constant on each such  $B_{\xi}$ . Secondly we will show that for each such  $B_{\xi}$  there is a sub-ball  $B'_{\xi} \subset B_{\xi}$  with  $\text{Vol}(B_{\xi}) = q_K \cdot \text{Vol}(B'_{\xi})$  and such that the second inequality of (3.5.5) holds for all  $y \in B'_{\xi}$ . These two facts together with the previous application of the induction hypothesis imply (3.5.6) and thus finish the proof for  $m=1$ . Fix  $B_{\xi} \subset U_{x,r}$  and write  $y = \xi + \gamma_{x,r} y' \in B_{\xi}$  for  $y' \in \mathcal{O}_K$ . By (3.5.2), (3.5.3), and (3.5.4), for each  $\ell$  and  $j$  there is a constant  $c_{\ell j} \in \mathbb{C}$  such that

$$f_{\ell j}(x, y) = c_{\ell j} \psi_K(b'_{\ell j} y'),$$

where we can take  $b'_{\ell j} = \gamma_{x,r} b_{x,r,\ell,i}$  for any  $i \in S_{\ell j}$ . This shows that  $|f_{\ell j}(x, \cdot)|_{\mathbb{C}}$  is constant on  $B_{\xi}$ . We now only have to construct  $B'_{\xi}$ . By renumbering, we can suppose that on  $B_{\xi}$ ,  $|f_{1,1}|_{\mathbb{C}}$  is maximal among the  $|f_{\ell j}|_{\mathbb{C}}$ , so that the middle expression of (3.5.5) is equal to  $|f_{1,1}|_{\mathbb{C}}$ . In particular, it suffices to choose  $B'_{\xi}$  such that  $|f_{1,1}|_{\mathbb{C}} \leq |f_1(x, y)|_{\mathbb{C}}$  for  $y \in B'_{\xi}$ . Now let  $\psi$  be the additive character of  $\mathbb{F}_{q_K}$  satisfying  $\psi_K(y') = \psi(\text{res}(y'))$  for  $y' \in \mathcal{O}_K$ . By (3.5.4), we have  $\text{res}(b'_{1j}) \neq \text{res}(b'_{1j'})$  for each  $j \neq j'$ , so we can apply Corollary 3.5.2 to

$$\tilde{f} : \mathbb{F}_{q_K} \rightarrow \mathbb{C} : \tilde{y} \mapsto \sum_j c_{1j} \psi(\text{res}(b'_{1j}) \cdot \tilde{y})$$

and get an  $\tilde{y}_0 \in \mathbb{F}_{q_K}$  with  $|c_{1,1}|_{\mathbb{C}} \leq |\tilde{f}(\tilde{y}_0)|_{\mathbb{C}}$ . Set  $B'_{\xi} := \{\xi + \gamma_{x,r} y' \mid y' \in \text{res}^{-1}(\tilde{y}_0)\}$ . Since  $f_1(x, y) = \tilde{f}(\text{res}(y'))$  and  $|f_{1,1}|_{\mathbb{C}} = |c_{1,1}|_{\mathbb{C}}$ , we are done.  $\square$

*Proof of Proposition 3.2.5 for  $m > 1$ .* Denote  $(y_1, \dots, y_{m-1})$  by  $\hat{y}$ . Apply the  $m=1$  case using  $(x, \hat{y})$  as parameters and  $y_m$  as the only  $y$ -variable. This yields in particular an integer  $d_1 > 0$ , a surjection  $\varphi_1 : U \rightarrow V_1$ , and a collection of functions  $G_{\ell i 1}$ . Now apply the induction hypothesis to the collection of functions  $G_{\ell i 1}$ , this time using  $\hat{y}$  as the  $y$ -variables, and the variables  $(x, \hat{y}, r) \in V_1$  as parameters. This yields an integer  $d_2$ , a surjection  $\varphi_2 : V_1 \rightarrow V_2$ , and a collection of functions  $G_{\ell i 2}$ . Now define  $\varphi$  as  $\varphi_2 \circ \varphi_1$  and consider the functions  $G_{\ell i} = G_{\ell i 2}$  on  $V_2$ . Since  $K$ -valued definable functions

like the  $h_{\ell i}$  can only depend piecewise trivially on variables running over  $\mathbb{Z}$  we are done, namely, the functions  $G_{\ell i}$  and  $\varphi$  satisfy the proposition with  $d = d_1 + d_2$ .  $\square$

*Proof of Theorems 3.2.2 and 3.2.3.* Let  $f$  be in  $\mathcal{C}^{\text{exp}}(X \times K^m)$  for some definable set  $X$  and write  $f$  as  $\sum_{i=1}^N G_i(\varphi(x, y))\psi_K(h_i(x, y))$  as in Proposition 3.2.5 with  $s = 1$ ,  $f_1 = f$ , and  $U = X \times K^m$ , so that in particular the  $h_i : X \times K^m \rightarrow K$  and  $\varphi : U \rightarrow V$  are definable functions, and the  $G_i(x, y)$  lie in  $\mathcal{C}^{\text{exp}}(V)$ . For each  $i$  let  $H_i$  be the function  $G_i \circ \varphi$ . Since  $K$ -valued functions can only depend piecewise trivially on  $\mathbb{Z}$ -variables, one has a natural isomorphism

$$\mathcal{C}^{\text{exp}}(V) \cong \mathcal{C}^{\text{exp}}(X) \otimes_{\mathcal{C}(X)} \mathcal{C}(V),$$

and thus, the  $H_i$  lie in  $\mathcal{C}^{\text{exp}}(X) \otimes_{\mathcal{C}(X)} \mathcal{C}(X \times K^m)$ . For these tensor products of algebras we use the natural inclusions  $\mathcal{C}(X) \subset \mathcal{C}(V)$ ,  $\mathcal{C}(X) \subset \mathcal{C}^{\text{exp}}(X)$ , and  $\mathcal{C}(X) \subset \mathcal{C}(X \times K^m)$ , which are inclusions of algebras.

It is clear that for any  $x \in X$ , if  $x \in \text{Iva}(H_i, X)$  for all  $i$ , then  $x \in \text{Iva}(f, X)$ . Vice versa, if  $f(x, \cdot)$  is identically zero, then  $H_i(x, \cdot)$  is zero on each set  $W_{x,r}$ , and since it is constant on each set  $U_{x,r}$ ,  $H_i(x, \cdot)$  is identically zero. Thus we just showed:

$$\text{Iva}(f, X) = \bigcap_i \text{Iva}(H_i, X).$$

A similar argument shows

$$\text{Bdd}(f, X) = \bigcap_i \text{Bdd}(H_i, X),$$

$$\text{Int}(f, X) = \bigcap_i \text{Int}(H_i, X),$$

where in the case of  $\text{Int}(f, X)$ , we use the inequality between the volumes of  $W_{x,r}$  and  $U_{x,r}$  given by Proposition 3.2.5.

Now we are done by applying Corollary 3.4.4 to each of the functions  $H_i$  and by combining the outcomes.  $\square$

For the proof of Theorem 3.2.1 we will apply Theorem 8.6.1 (1) of [9], which has stringent integrability conditions. These conditions can be satisfied by the last part of the statement of Theorem 3.2.3.

*Proof of Theorem 3.2.1.* Let  $g$  be given by Theorem 3.2.3. Since, by the same theorem,  $g$  is a finite sum of terms of the form

$$f_0 \psi_K(f_1)$$

with  $f_1 : X \times K^m \rightarrow K$  definable and  $f_0 \in \mathcal{C}(X \times K^m)$  satisfying  $\text{Int}(f_0, X) = X$ , the function  $g$  falls under the scope of Theorem 8.6.1 (1) of [9], which yields the desired conclusion.  $\square$

## 4. TRANSFER PRINCIPLES FOR INTEGRABILITY AND BOUNDEDNESS

**4.1. Notation.** Let  $\mathcal{O}$  be a ring of integers of a number field. We will use the first order language of Denef-Pas with coefficients in  $\mathcal{O}[[t]]$ , and denote it by  $\mathcal{L}_{\text{DP}}$ . By definable we will from now on mean  $\mathcal{L}_{\text{DP}}$ -definable, without using other coefficients than those from  $\mathcal{O}[[t]]$ . Recall that  $\mathcal{L}_{\text{DP}}$  has three sorts: the valued field, the residue field, and the value group. The language  $\mathcal{L}_{\text{DP}}$  has as symbols the usual logical symbols, the language of rings  $(+, -, \cdot, 0, 1)$  with coefficients from  $\mathcal{O}[[t]]$  for the valued field, another copy of the language of rings for the residue field, the Presburger language  $(+, -, \leq, \{\cdot \equiv \cdot \bmod n\}_{n>1}, 0, 1)$  for the value group, the symbol  $\text{ord}$  for the valuation map on the nonzero elements of the valued field, and the symbol  $\overline{\text{ac}}$  for an angular component map. All structures for  $\mathcal{L}_{\text{DP}}$  that we will consider are triples  $(L, k_L, \mathbb{Z})$  with  $L$  a complete discretely valued field,  $\mathcal{O}_L$  its valuation ring with residue field  $k_L$ , and value group identified with  $\mathbb{Z}$ , together with the information of how the symbols of  $\mathcal{L}_{\text{DP}}$  are interpreted in this triple. To fix the meaning of the symbols of  $\mathcal{L}_{\text{DP}}$  one fixes a ring homomorphism  $\lambda_{\mathcal{O},L} : \mathcal{O}[[t]] \rightarrow \mathcal{O}_L$  respecting 1 and sending  $t$  to a uniformizer  $\varpi$  of  $\mathcal{O}_L$ . If one fixes such  $\lambda_{\mathcal{O},L}$  then all the symbols of  $\mathcal{L}_{\text{DP}}$  have a unique meaning where we require that  $\overline{\text{ac}} : L \rightarrow k_L$  is the unique multiplicative map which extends the projection  $\mathcal{O}_L^\times \rightarrow k_L^\times$  and sends  $\varpi$  to 1; it is given by

$$\overline{\text{ac}} : L \rightarrow k_L : \begin{cases} x\varpi^{-\text{ord } x} \bmod (\varpi) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

(the other symbols have their natural meaning). Note that, by the completeness of  $L$ , a ring homomorphism  $\mathcal{O} \rightarrow \mathcal{O}_L$  and the choice of the uniformizer  $\varpi$  of  $\mathcal{O}_L$  determine a ring homomorphism  $\mathcal{O}[[t]] \rightarrow \mathcal{O}_L$  sending  $t$  to  $\varpi$ .

Let  $\mathcal{A}_{\mathcal{O}}$  be the collection of non-Archimedean local fields  $K$  of characteristic zero with a ring homomorphism  $\mathcal{O} \rightarrow K$  and a uniformizer  $\varpi_K$  of  $\mathcal{O}_K$ . Let  $\mathcal{B}_{\mathcal{O}}$  be the collection of all local fields  $K$  of positive characteristic with a ring homomorphism  $\mathcal{O} \rightarrow K$  and a uniformizer  $\varpi_K$  of  $\mathcal{O}_K$ . Let  $\mathcal{C}_{\mathcal{O}}$  be the union of  $\mathcal{A}_{\mathcal{O}}$  and  $\mathcal{B}_{\mathcal{O}}$ . For an integer  $M > 0$ , denote by  $\mathcal{A}_{\mathcal{O},M}$ ,  $\mathcal{B}_{\mathcal{O},M}$ , resp.  $\mathcal{C}_{\mathcal{O},M}$  those fields in  $\mathcal{A}_{\mathcal{O}}$ ,  $\mathcal{B}_{\mathcal{O}}$ , resp.  $\mathcal{C}_{\mathcal{O}}$  that have residue characteristic larger than  $M$ . For  $K$  in  $\mathcal{C}_{\mathcal{O}}$ , write  $\mathcal{M}_K$  for the maximal ideal of  $\mathcal{O}_K$ ,  $k_K$  for the residue field and  $q_K$  for the number of elements of  $k_K$ . For  $x \in \mathcal{O}_K$ , denote by  $\overline{x} \in k_K$  the reduction of  $x$  modulo  $(\varpi_K)$ .

For  $K \in \mathcal{C}_{\mathcal{O}}$ , write  $\mathcal{D}_K$  for the collection of additive characters  $\psi : K \rightarrow \mathbb{C}^\times$  which are trivial on the maximal ideal  $\mathcal{M}_K$  and which coincide on  $\mathcal{O}_K$  with the character sending  $x \in \mathcal{O}_K$  to

$$\exp\left(\frac{2\pi i}{p} \text{Tr}_{k_K}(\overline{x})\right),$$

where  $\text{Tr}_{k_K}$  is the trace of  $k_K$  over its prime subfield and  $p$  is the characteristic of  $k_K$ . Note that there is no restriction in only considering additive characters

lying in  $\mathcal{D}_K$ , since, in our set-up, all other additive characters on  $K$  can appear naturally by using a parameter over the valued field.

For any  $K \in \mathcal{C}_{\mathcal{O}}$ , the measure we put on  $K^n \times k_K^m \times \mathbb{Z}^r$  is the product measure of the Haar measure on  $K^n$  normalized so that  $\mathcal{O}_K^n$  has measure 1 with the discrete measure (the counting measure) on  $k_K^m \times \mathbb{Z}^r$ . Likewise, put on  $K^n \times k_K^m \times \mathbb{Z}^r$  the product topology of the valuation topology on  $K^n$  with the discrete topology on  $k_K^m \times \mathbb{Z}^r$ .

**4.2. The motivic setting.** We recall the settings from [8] and [9], where we will write  $\mathcal{Q}$  instead of  $K_0(\text{RDef})$ .

**4.2.1. Definable subassignments.** For any field  $k$  of characteristic zero, we consider the Laurent series field  $k((t))$  over  $k$  with the uniformizer  $t$  and the corresponding angular component map and discrete valuation.

Any  $\mathcal{L}_{\text{DP}}$ -formula  $\varphi$  in  $m$  free valued field variables,  $n$  free residue field variables, and  $r$  free value group variables, and any field  $k$  of characteristic zero which contains our fixed ring of integers  $\mathcal{O}$  as a subring gives rise to a subset of

$$k((t))^m \times k^n \times \mathbb{Z}^r$$

consisting of the points satisfying  $\varphi$ , which can be written symbolically as follows:

$$\{(x, y, z) \in k((t))^m \times k^n \times \mathbb{Z}^r \mid \varphi(x, y, z)\};$$

this subset is denoted by  $\varphi_{k((t))}$ .

By a definable subassignment we mean the map  $X$  which sends  $k$  to  $X(k) := \varphi_{k((t))}$  for some  $\mathcal{L}_{\text{DP}}$ -formula  $\varphi$ , where  $k$  runs over characteristic zero fields which contain  $\mathcal{O}$  as a subring. Denote by  $h$  the definable subassignment which sends  $k$  to the singleton  $\{0\}$ , also written as  $k((t))^0 \times k^0 \times \mathbb{Z}^0$ . (For readers familiar with the language of model theory, note that two formulas  $\varphi$  and  $\varphi'$  yield the same definable subassignment iff they are equivalent modulo the theory of Henselian valued fields of characteristic  $(0, 0)$  with value group  $\cong \mathbb{Z}$  and whose residue fields contain  $\mathcal{O}$  as a subring.)

For any definable subassignment  $X$ , and for nonnegative integers  $m, n, r$ , write  $X[m, n, r]$  for the definable subassignment sending  $k$  to

$$X(k) \times k((t))^m \times k^n \times \mathbb{Z}^r.$$

For example,  $h[m, n, r]$  sends  $k$  to  $k((t))^m \times k^n \times \mathbb{Z}^r$ .

A point on a definable subassignment  $X$  consists of a pair  $(x, k)$  with  $k$  a characteristic zero field having  $\mathcal{O}$  as a subring and with  $x$  an element of  $X(k)$ . We write  $|X|$  for the collection of all points that lie on  $X$ . (We leave it to the reader to choose whether to consider  $|X|$  as an actual class or to work in a fixed large universe.)

The usual set-theoretic operations make sense for definable subassignments. If  $X(k) \subset Y(k)$  for each  $k$ , then we also call  $X$  a definable subassignment of  $Y$ .

By a definable morphism  $f : X \rightarrow Y$  between definable subassignments  $X$  and  $Y$  is meant a definable subassignment  $G \subset X \times Y$  such that  $G(k)$  is



the graph of a function from  $X(k)$  to  $Y(k)$  for each  $k$  and one calls  $G$  the graph of  $f$ . We write  $f_k$  for the function from  $X(k)$  to  $Y(k)$  with the graph  $G(k)$ .

Write  $\text{Def}$  for the category of definable subassignments with definable morphisms as morphisms.

**4.2.2. Definable subassignments and local fields.** We have seen that behind a definable subassignment  $X$  lies an  $\mathcal{L}_{\text{DP}}$ -formula  $\varphi$  which describes the sets  $X(k)$ . Clearly such a formula  $\varphi$  corresponding to  $X$  is not unique. However, if we fix such a  $\varphi$  for  $X$ , which we call fixing a representative of  $X$ , then for each  $K \in \mathcal{C}_{\mathcal{O}}$ , we can consider the subset  $\varphi_K$  of  $K^m \times k_K^n \times \mathbb{Z}^r$  consisting of the points satisfying  $\varphi$ , where  $X$  is a definable subassignment of  $h[m, n, r]$ . Indeed, all the symbols of  $\mathcal{L}_{\text{DP}}$  can be interpreted in the three sorts  $K, k_K, \mathbb{Z}$ , where elements of  $\mathcal{O}[[t]]$  are interpreted in  $K$  via the ring homomorphism  $\mathcal{O}[[t]] \rightarrow K$  coming from the ring homomorphism  $\mathcal{O} \rightarrow K$  and sending  $t$  to the uniformizer  $\varpi_K$ .

For all motivic objects in this paper, we will make a link with objects (usually sets and functions) on local fields. In particular, given a definable subassignment  $X$ , we will often implicitly fix a representative  $\varphi$  and write  $X_K$  instead of  $\varphi_K$  where  $K$  in  $\mathcal{C}_{\mathcal{O}, M}$  for sufficiently large  $M$ . Although  $X_K$  depends on the choice of  $\varphi$ , we have the following phenomenon:

For any two representatives  $\varphi$  and  $\varphi'$  of a definable subassignment  $X$ , there exists  $M > 0$  such that  $\varphi_K = \varphi'_K$  for all  $K$  in  $\mathcal{C}_{\mathcal{O}, M}$ .

*Remark 4.2.3.* The operation of taking representatives in this context is similar to the notion of taking a model over  $\mathbb{Z}$  of a variety defined over  $\mathbb{Q}$  in the context of algebraic geometry, as one typically does for counting the number of rational points over finite fields.

Similarly, any definable morphism  $f : X \rightarrow Y$  between definable subassignments gives rise, up to fixing a formula  $\gamma$  corresponding to the graph of  $f$ , to a function

$$f_K : X_K \mapsto Y_K,$$

whose graph is  $\gamma_K$  for any  $K$  in  $\mathcal{C}_{\mathcal{O}, M}$  with  $M$  sufficiently large. (If the characteristic of the residue field of  $K$  is small, then  $\gamma_K$  might not define the graph of a function.)

**4.2.4. Constructible motivic functions.** Recall that  $h[0, 0, 1]$  can be identified with  $\mathbb{Z}$ , since  $h[0, 0, 1](k) = \mathbb{Z}$  for all  $k$ . Let  $X$  be in  $\text{Def}$ , that is, let  $X$  be a definable subassignment. A definable morphism  $\alpha : X \rightarrow h[0, 0, 1]$  gives rise to a function  $|X| \rightarrow \mathbb{Z}$  (also denoted by  $\alpha$ ) sending a point  $(x, k)$  on  $X$  to  $\alpha_k(x)$ . Likewise, such  $\alpha$  gives rise to the function  $\mathbb{L}^\alpha$  from  $|X|$  to  $\mathbb{A}$  which sends a point  $(x, k)$  on  $X$  to  $\mathbb{L}^{\alpha_k(x)}$ , and where  $\mathbb{A}$  is as in Section 2.2.

Following [8], we define the ring  $\mathcal{P}(X)$  of constructible Presburger functions on  $X$  as the subring of the ring of functions  $|X| \rightarrow \mathbb{A}$  generated by

- (1) all constant functions into  $\mathbb{A}$ ,
- (2) all functions  $\alpha : |X| \rightarrow \mathbb{Z}$  with  $\alpha : X \rightarrow h[0, 0, 1]$  a definable morphism,
- (3) all functions of the form  $\mathbb{L}^\beta$  with  $\beta : X \rightarrow h[0, 0, 1]$  a definable morphism.

Note that although  $|X|$  is not a set,  $\mathcal{P}(X)$  can be regarded as a set since it has not too many generators.

For  $Y$  a definable subassignment of  $X$ , write  $\mathbf{1}_Y$  for the characteristic function of  $Y$ , sending a point  $(x, k)$  on  $X$  to 1 if it lies on  $Y$  and to zero otherwise.

Define the group  $\mathcal{Q}(X)$  as the quotient of the free abelian group over symbols  $[Y]$  with  $Y$  a subassignment of  $X[0, m, 0]$  for some  $m \geq 0$ , by the following scissor relations.

$$(4.2.1) \quad [Y] = [Y']$$

if there exists a definable isomorphism  $Y \rightarrow Y'$  which commutes with the projections  $Y \rightarrow X$  and  $Y' \rightarrow X$ .

$$(4.2.2) \quad [Y_1 \cup Y_2] + [Y_1 \cap Y_2] = [Y_1] + [Y_2]$$

for  $Y_1$  and  $Y_2$  definable subassignments of a common  $X[0, m, 0]$  for some  $m$ .

We will still write  $[Y]$  for the class of  $[Y]$  in  $\mathcal{Q}(X)$  for  $Y \subset X[0, m, 0]$ . In [8], the longer notation  $K_0(\text{RDef}_X)$  is used instead of  $\mathcal{Q}(X)$ . Denote by  $\mathcal{P}^0(X)$  the subring of  $\mathcal{P}(X)$  generated by the characteristic functions  $\mathbf{1}_Y$  for all definable subassignments  $Y$  of  $X$  and by the constant function  $\mathbb{L}$ . Using the canonical ring morphism  $\mathcal{P}^0(X) \rightarrow \mathcal{Q}(X)$ , sending  $\mathbf{1}_Y$  to  $[Y]$  and  $\mathbb{L}$  to the class of  $X[0, 1, 0]$ , we define the ring  $\mathcal{C}(X)$  as

$$\mathcal{P}(X) \otimes_{\mathcal{P}^0(X)} \mathcal{Q}(X).$$

Elements of  $\mathcal{C}(X)$  are called constructible motivic functions on  $X$ .

Let  $F$  be a function in  $\mathcal{C}(X[m, 0, 0])$  for some  $m \geq 0$ . Using notation from [8] Section 13.2, we say that  $F$  is (motivically)  $X$ -integrable if and only if its class in  $C^m(X[m, 0, 0] \rightarrow X)$  lies in  $\text{I}_X C(X[m, 0, 0] \rightarrow X)$ , where  $X[m, 0, 0] \rightarrow X$  is the projection. Although we refer to [8] for precise definitions, and although we do not need this notion for the transfer principles, let us give some intuitive explanation of motivic  $X$ -integrability. The condition for  $F$  to be (motivically)  $X$ -integrable is a strong uniform form of the condition that for all  $K$  in  $\mathcal{C}_{O, M}$  with  $M$  sufficiently large,  $F_K(x, \cdot)$  is integrable over  $K^m$  with respect to the Haar measure, for each  $x \in X_K$ . This motivic condition is defined, via cell decomposition techniques, in terms of  $X$ -integrability of functions  $G$  in  $\mathcal{P}(X \times \mathbb{Z}^m)$ . A function  $G$  in  $\mathcal{P}(X \times \mathbb{Z}^m)$  is considered  $X$ -integrable if and only if for each  $(x, k) \in |X|$ , the family

$$(G_k(x, z)(q))_{z \in \mathbb{Z}^m}$$

is summable (in the classical sense), for each real  $q > 1$ , where we use evaluation at  $\mathbb{L} = q$  as in (2.2.1).

4.2.5. *Constructible motivic functions and local fields.* Each  $f$  in  $\mathcal{P}(X)$ , with  $X$  a definable subassignment, can be written as a finite sum of terms of the form  $a\mathbb{L}^\beta \prod_{i=1}^\ell \alpha_i$  with  $a \in \mathbb{A}$ , and the  $\alpha_i$  and  $\beta$  definable morphisms from  $X$  to  $h[0, 0, 1] = \mathbb{Z}$ . Let us take representatives  $\alpha'_i$  and  $\beta'$  of the  $\alpha_i$  and  $\beta$ , that is, the  $\mathcal{L}_{\text{DP}}$ -formulas describing the graphs. We have seen in Section 4.2.2 that, for  $K$  in  $\mathcal{C}_{\mathcal{O}, M}$  with  $M$  sufficiently large,  $\alpha'_{iK}$  and  $\beta'_K$  are the graphs of functions from  $X_K$  to  $\mathbb{Z}$  and we have denoted these functions by  $\alpha_{iK}$  and  $\beta_K$ . Now we extend this notation to elements  $f$  of  $\mathcal{P}(X)$ , where we write  $f_K$  for the function sending  $x \in X_K$  to

$$\sum_j a_j(q_K) q_K^{\beta_{jK}(x)} \prod_{i=1}^{\ell_j} \alpha_{ijK}(x),$$

whenever  $f$  equals

$$\sum_j a_j \mathbb{L}^{\beta_j} \prod_{i=1}^{\ell_j} \alpha_{ij},$$

where  $a_j \in \mathbb{A}$ ,  $a_j(q_K)$  is the evaluation of  $a_j$  at  $\mathbb{L} = q_K$  as in (2.2.1), and the  $\alpha_{ij}$  and  $\beta_j$  are definable morphisms from  $X$  to  $\mathbb{Z}$ . In a similar sense as in Section 4.2.2, the function  $f_K : X_K \rightarrow \mathbb{Q}$  is independent of the choice of the representatives for the  $\alpha_{ij}$  and  $\beta_i$  whenever  $K$  is in  $\mathcal{C}_{\mathcal{O}, M}$  with  $M$  sufficiently large.

Likewise, since each  $g$  in  $\mathcal{Q}(X)$  can be written as  $[Y] - [Z]$  for some definable subassignments  $Y \subset X[0, n, 0]$  and  $Z \subset X[0, n', 0]$ , by taking representatives, one can consider  $Y_K$  and  $Z_K$  for  $K$  in  $\mathcal{C}_{\mathcal{O}, M}$  with  $M$  sufficiently large, and we denote by  $g_K$  the function on  $X_K$  sending  $x \in X_K$  to

$$\sharp Y_x - \sharp Z_x,$$

where  $Y_x$  is the (finite) set  $\{r \in k_K^n \mid (x, r) \in Y_K\}$  of size  $\sharp Y_x$  and likewise for  $Z_x$ .

Since for  $f \in \mathcal{P}^0(X)$  and  $f'$  its image in  $\mathcal{Q}(X)$  one has  $f_K = f'_K$  for all  $K$  in  $\mathcal{C}_{\mathcal{O}, M}$  with  $M$  sufficiently large, one can define for  $F$  in  $\mathcal{C}(X)$  and for  $K$  in  $\mathcal{C}_{\mathcal{O}, M}$  with  $M$  sufficiently large, the function  $F_K$  as

$$F_K : X_K \rightarrow \mathbb{Q} : x \mapsto \sum_i a_{iK}(x) b_{iK}(x)$$

whenever  $F = \sum_i a_i \otimes b_i$  with  $a_i \in \mathcal{P}(X)$  and  $b_i \in \mathcal{Q}(X)$ . In our usual sense, this is independent of the choice of representatives when  $K$  is in  $\mathcal{C}_{\mathcal{O}, M}$  with  $M$  sufficiently large.

4.2.6. *Motivic exponential functions.* Let  $X$  be in Def. We consider the category  $Q_X^{\text{exp}}$  whose objects are the triples  $(Y, \xi, g)$  with  $Y$  a definable subassignment of  $X[0, n, 0]$  for some  $n \geq 0$ , and  $\xi : Y \rightarrow h[0, 1, 0]$  and  $g : Y \rightarrow h[1, 0, 0]$  definable morphisms. A morphism  $(Y', \xi', g') \rightarrow (Y, \xi, g)$  in  $Q_X^{\text{exp}}$  is a definable morphism  $h : Y' \rightarrow Y$  which makes a commutative diagram with the projections to  $X$  and such that  $\xi' = \xi \circ h$  and  $g' = g \circ h$ .

To the category  $Q_X^{\text{exp}}$  one assigns a ring  $\mathcal{Q}^{\text{exp}}(X)$  defined as follows. As an abelian group it is the quotient of the free abelian group over the symbols  $[Y, \xi, g]$  with  $(Y, \xi, g)$  in  $Q_X^{\text{exp}}$  by the following four relations

$$(4.2.3) \quad [Y, \xi, g] = [Y', \xi', g']$$

for  $(Y, \xi, g)$  isomorphic to  $(Y', \xi', g')$ ,

$$(4.2.4) \quad [Y \cup Y', \xi, g] + [Y \cap Y', \xi|_{Y \cap Y'}, g|_{Y \cap Y'}] \\ = [Y, \xi|_Y, g|_Y] + [Y', \xi|_{Y'}, g|_{Y'}]$$

for  $Y$  and  $Y'$  definable subassignments of some common  $X[0, n, 0]$  for some  $n \geq 0$  and  $\xi, g$  defined on  $Y \cup Y'$ ,

$$(4.2.5) \quad [Y, \xi, g + g'] = [Y, \xi + \bar{g}', g]$$

for  $g' : Y \rightarrow h[1, 0, 0]$  a definable morphism with  $\text{ord}(g'(y)) \geq 0$  for all  $y$  in  $Y$  and  $\bar{g}'$  the reduction of  $g'$  modulo the maximal ideal, and

$$(4.2.6) \quad [Y[0, 1, 0], \xi + p, g] = 0$$

when  $p : Y[0, 1, 0] \rightarrow h[0, 1, 0]$  is the projection and when  $g$  and  $\xi$  factorize through the projection  $Y[0, 1, 0] \rightarrow Y$ .

**Lemma 4.2.7** ([9]). *We may endow  $\mathcal{Q}^{\text{exp}}(X)$  with a ring structure by setting*

$$[Y, \xi, g] \cdot [Y', \xi', g'] = [Y \otimes_X Y', \xi \circ p_Y + \xi' \circ p_{Y'}, g \circ p_Y + g' \circ p_{Y'}],$$

where  $Y \otimes_X Y'$  is the fiber product of  $Y$  and  $Y'$ ,  $p_Y$  the projection to  $Y$ , and  $p_{Y'}$  the projection to  $Y'$ .

By [9] there is a natural injection of rings  $\mathcal{Q}(X) \rightarrow \mathcal{Q}^{\text{exp}}(X)$  sending  $[Y]$  to  $[Y, 0, 0]$ . Hence, we may define the ring  $\mathcal{C}^{\text{exp}}(X)$  of motivic exponential functions by

$$(4.2.7) \quad \mathcal{C}^{\text{exp}}(X) := \mathcal{C}(X) \otimes_{\mathcal{Q}(X)} \mathcal{Q}^{\text{exp}}(X).$$

*Remark 4.2.8.* Note that in [9],  $\mathcal{Q}(X)$  is denoted by  $K_0(\text{RDef}_X)$ ,  $\mathcal{Q}^{\text{exp}}(X)$  is denoted by  $K_0(\text{RDef}_X^{\text{exp}})$ , and  $\mathcal{C}^{\text{exp}}(X)$  is denoted by  $\mathcal{C}(X)^{\text{exp}}$ .

Let  $F$  be a motivic exponential function in  $\mathcal{C}^{\text{exp}}(X[m, 0, 0])$  for some  $m \geq 0$ . Using notation from [9], we say that  $F$  is (motivically)  $X$ -integrable if and only if its class in  $C^m(X[m, 0, 0] \rightarrow X)^{\text{exp}}$  lies in  $I_X C(X[m, 0, 0] \rightarrow X)^{\text{exp}}$ , where  $X[m, 0, 0] \rightarrow X$  is the projection. Again the notion of  $X$ -integrability boils down to the classical of summability of countable families via cell decomposition techniques, see the end of Section 4.2.4.

**4.2.9. Motivic exponential functions and local fields.** In this section we explain, following [9], how to find actual functions  $f_{K,\psi} : X_K \rightarrow \mathbb{C}$  for  $f \in \mathcal{C}^{\text{exp}}(X)$ ,  $K$  in  $\mathcal{C}_{\mathcal{O},M}$  with  $M$  sufficiently large,  $\psi \in \mathcal{D}_K$ , and  $X$  a definable subassignment. For  $f$  in  $\mathcal{C}(X)$  this was explained in Section 4.2.5. Take  $G = [Y, \xi, g]$  in  $\mathcal{Q}^{\text{exp}}(X)$ , with  $Y \subset X[0, n, 0]$ , take representatives of  $Y$ ,  $\xi$ , and  $g$ , and let  $K$  be in  $\mathcal{C}_{\mathcal{O},M}$  with  $M$  sufficiently large, so that  $\xi_K$  and  $g_K$  are functions from  $Y_K$  to  $k_K$ , resp. to  $K$ . Then we define  $G_{K,\psi}$  as the function sending  $x \in X_K$  to the exponential sum

$$\sum_{r \in Y_x} \psi(\xi_K(x, r) + g_K(x, r)),$$

which is well defined since  $\psi$  is trivial on  $\mathcal{M}_K$ , and since  $\xi_K(x, r)$  can be considered as an element of  $\mathcal{O}_K \bmod \mathcal{M}_K$ . Finally, for  $f \in \mathcal{C}^{\text{exp}}(X)$ ,  $K$  in  $\mathcal{C}_{\mathcal{O},M}$  with  $M$  sufficiently large, and  $\psi \in \mathcal{D}_K$ , we define  $f_{K,\psi}$  by

$$f_{K,\psi} : X_K \rightarrow \mathbb{C} : x \mapsto \sum_i a_{iK}(x) b_{iK,\psi}(x)$$

whenever  $f = \sum_i a_i \otimes b_i$  with  $a_i \in \mathcal{C}(X)$  and  $b_i \in \mathcal{Q}^{\text{exp}}(X)$ .

We recapitulate how  $f_{K,\psi}$  is independent of the choice of representatives for  $K$  in  $\mathcal{C}_{\mathcal{O},M}$  with  $M$  sufficiently large. For any two different collections  $C_1$  and  $C_2$  of representatives of the  $\mathcal{L}_{\text{DP}}$ -formulas that go into the description of  $f$ , there exists  $M'$  such that for all  $K$  in  $\mathcal{C}_{\mathcal{O},M'}$  and all  $\psi \in \mathcal{D}_K$ , one has that  $f_{K,\psi}$  is independent of the choice of one of the  $C_i$ .

**4.3. The constructible setting.** We find motivic analogues of our thematic results, namely the analogues of the  $p$ -adic Theorems 3.1.1, 3.1.2, and 3.1.4.

**Theorem 4.3.1** (Integration). *Let  $f$  be in  $\mathcal{C}(X[m, 0, 0])$  for some  $m \geq 0$  and some definable subassignment  $X$ . Then there exists  $g$  in  $\mathcal{C}(X)$  such that for all  $K$  in  $\mathcal{C}_{\mathcal{O}}$  with large enough residue field characteristic,*

$$g_K(x) = \int_{y \in K^m} f_K(x, y),$$

*whenever  $x \in \text{Int}(f_K, X_K)$ .*

The special case of the above theorem when  $f$  is motivically  $X$ -integrable follows from [8] and [9].

**Theorem 4.3.2** (Correspondences of loci). *Let  $f$  be in  $\mathcal{C}(X[m, 0, 0])$  for some definable subassignment  $X$  and some  $m \geq 0$ . Then there exists  $h_1, h_2, h_3 \in \mathcal{C}(X)$  such that, for all  $K$  in  $\mathcal{C}_{\mathcal{O}}$  with large enough residue field characteristic, the zero locus of  $h_{iK}$  in  $X_K$  equals  $\text{Int}(f_K, X_K)$ , resp.  $\text{Bdd}(f_K, X_K)$  resp.  $\text{Iva}(f_K, X_K)$ , when  $i$  is 1, 2, or 3, respectively.*

The analogues of Corollaries 3.1.3 and 3.2.4 in the motivic context also hold by the same argument, but this is left to the reader (see also Corollary 3.2.4 below).

**Theorem 4.3.3** (Interpolation). *Let  $f$  be in  $\mathcal{C}(X[m, 0, 0])$  for some  $m \geq 0$  and some definable subassignment  $X$ . Then there exists  $g$  in  $\mathcal{C}(X[m, 0, 0])$  such that the following hold for  $K$  in  $\mathcal{C}_{\mathcal{O}}$  with large enough residue field characteristic.*

- (1)  $f_K(x, y) = g_K(x, y)$  whenever  $x \in \text{Int}(f_K, X_K)$ , and for all  $y \in K^m$ ,
- (2)  $\text{Int}(g_K, X_K) = X_K$ ,
- (3)  $g$  is (motivically)  $X$ -integrable.

**4.4. The exponential setting and transfer principles.** The following two theorems constitute the general transfer principles and the main results of this paper.

**Theorem 4.4.1** (Transfer principle for integrability). *Let  $f$  be in  $\mathcal{C}^{\text{exp}}(X[m, 0, 0])$  for some  $m \geq 0$  and some definable subassignment  $X$ . Then, for all  $K \in \mathcal{C}_{\mathcal{O}, M}$  for some large  $M$ , the truth of each of the following statements depends only on the (isomorphism class of the) residue field of  $K$ .*

- (1) For all  $x \in X_K$  and for all  $\psi \in \mathcal{D}_K$ , the function  $f_{K, \psi}(x, \cdot)$  is integrable over  $K^m$ , that is,  $\text{Int}(X_K, f_{K, \psi}) = X_K$  for all  $\psi \in \mathcal{D}_K$ .
- (2) For all  $x \in X_K$  and for all  $\psi \in \mathcal{D}_K$ , the function  $f_{K, \psi}(x, \cdot)$  is **locally** integrable on  $K^m$ .

**Theorem 4.4.2** (Transfer principle for boundedness). *Let  $f$  be in  $\mathcal{C}^{\text{exp}}(X[m, 0, 0])$  for some  $m \geq 0$  and some definable subassignment  $X$ . Then, for all  $K \in \mathcal{C}_{\mathcal{O}, M}$  for some large  $M$ , the truth of each of the following statements depends only on the (isomorphism class of the) residue field of  $K$ .*

- (1) For all  $x \in X_K$  and for all  $\psi \in \mathcal{D}_K$ , the function  $f_{K, \psi}(x, \cdot)$  is bounded on  $K^m$ .
- (2) For all  $x \in X_K$  and for all  $\psi \in \mathcal{D}_K$ , the function  $f_{K, \psi}(x, \cdot)$  is **locally** bounded on  $K^m$ .

The transfer principles will follow from motivic analogues of our common theme results, which we now state in our final, exponential setting.

**Theorem 4.4.3** (Integration). *Let  $f$  be in  $\mathcal{C}^{\text{exp}}(X[m, 0, 0])$  for some  $m \geq 0$  and some definable subassignment  $X$ . Then there exists  $g$  in  $\mathcal{C}^{\text{exp}}(X)$  such that the following holds for all  $K$  in  $\mathcal{C}_{\mathcal{O}}$  with large enough residue field characteristic, and for all  $\psi \in \mathcal{D}_K$ ,*

$$g_{K, \psi}(x) = \int_{y \in K^m} f_{K, \psi}(x, y),$$

whenever  $x \in \text{Int}(f_{K, \psi}, X_K)$ .

**Theorem 4.4.4** (Correspondences of loci). *Let  $f$  be in  $\mathcal{C}^{\text{exp}}(X[m, 0, 0])$  for some definable subassignment  $X$  and some  $m \geq 0$ . Then there exist  $h_1, h_2, h_3 \in \mathcal{C}^{\text{exp}}(X)$  such that, for all  $K$  in  $\mathcal{C}_{\mathcal{O}}$  with large enough residue field characteristic and for each  $\psi \in \mathcal{D}_K$ , the zero locus of  $h_{iK, \psi}$  in  $X_K$  equals respectively  $\text{Int}(X_K, f_{K, \psi})$ ,  $\text{Bdd}(X_K, f_{K, \psi})$ , and  $\text{Iva}(X_K, f_{K, \psi})$ , for  $i = 1, 2$ , or  $3$  respectively.*

Theorem 4.4.4 implies the following corollary by the same reasoning as for Corollary 3.1.3.

**Corollary 4.4.5.** *Let  $f$  be in  $\mathcal{C}^{\exp}(X[m, 0, 0])$  for some definable subassignment  $X$  and some  $m \geq 0$ . Then there exist functions  $h_1$  and  $h_2$  in  $\mathcal{C}^{\exp}(X)$  such that, for all  $K$  in  $\mathcal{C}_{\mathcal{O}}$  with large enough residue field characteristic and for each  $\psi \in \mathcal{D}_K$ , the zero locus of  $h_{1K,\psi}$  in  $X_K$  equals*

$$\{x \in X_K \mid f_{K,\psi}(x, \cdot) \text{ is locally integrable on } K^m\},$$

*and the zero locus of  $h_{2K,\psi}$  in  $X_K$  equals*

$$\{x \in X_K \mid f_{K,\psi}(x, \cdot) \text{ is locally bounded on } K^m\}.$$

**Theorem 4.4.6** (Interpolation). *Let  $f$  be in  $\mathcal{C}^{\exp}(X[m, 0, 0])$  for some definable subassignment  $X$  and some  $m \geq 0$ . Then there exist  $g$  in  $\mathcal{C}^{\exp}(X[m, 0, 0])$  and  $M > 0$  such that for all  $K$  in  $\mathcal{C}_{\mathcal{O},M}$  and all  $\psi \in \mathcal{D}_K$  one has*

- (1)  $f_{K,\psi}(x, y) = g_{K,\psi}(x, y)$  whenever  $x$  lies in  $\text{Int}(X_K, f_{K,\psi})$ ,
- (2)  $\text{Int}(X_K, g_{K,\psi}) = X_K$ ,
- (3)  $g$  is (motivically)  $X$ -integrable.

*Remark 4.4.7.* By standard techniques of motivic integration, all the above results (4.4.1 to 4.4.6, and also the ones in the previous sections) imply the corresponding results where  $X[m, 0, 0]$  is replaced by an arbitrary subassignment  $U \subset X[m, 0, 0]$ .

**4.5. Proofs of the motivic results.** We begin with some preliminaries.

**Definition 4.5.1.** Consider a definable subassignment  $X$ . A residual parametrization of  $X$  is by definition a definable isomorphism over  $X$  of the form

$$g : X \rightarrow X_{\text{par}} \subset X[0, m, 0]$$

for some  $m \geq 0$ . For  $F : X \rightarrow Y$  a definable morphism, write  $F_{\text{par}}$  for the corresponding definable morphism  $F \circ g^{-1} : X_{\text{par}} \rightarrow Y$ . Likewise, given  $f \in \mathcal{C}^{\exp}(X)$ , write  $f_{\text{par}}$  for the natural corresponding function in  $\mathcal{C}^{\exp}(X_{\text{par}})$ , and so on.

Up to residual parametrization (i.e., up to replacing  $X$  by  $X_{\text{par}}$  for a well-chosen residual parametrization), all results of Section 3.3 go through in a uniform way, as follows.

**Definition 4.5.2** (Uniform cells). Consider  $A \subset \Lambda[1, 0, 0]$  for some definable subassignment  $\Lambda$ . Then  $A$  is called a uniform 1-cell, resp. a uniform 0-cell, over  $\Lambda$  if there exists  $M > 0$  such that for all  $K$  in  $\mathcal{A}_{\mathcal{O},M}$  one has that  $A_K$  is a  $p$ -adic 1-cell, resp. a  $p$ -adic 0-cell, over  $\Lambda_K$ .

The next theorem follows from Denef-Pas cell decomposition [23] and the results of [7], Section 6.

**Theorem 4.5.3** (Uniform version of Theorem 3.3.2 and Proposition 3.3.5). *Consider  $X \subset \Lambda[1, 0, 0]$  with  $\Lambda$  and  $X$  definable subassignments and let  $f_j : X \rightarrow h[0, 0, 1]$  and  $F : X \rightarrow h[1, 0, 0]$  be definable morphisms. Then,*

up to replacing  $X$  by  $X_{\text{par}}$  and replacing the  $f_i$  and  $F$  correspondingly by  $(f_i)_{\text{par}}$  and  $F_{\text{par}}$  for a well-chosen residual parametrization of  $X$ , the following holds. There exist  $M > 0$  and a finite partition of  $X$  into definable subassignments  $A$  such that for all  $K$  in  $\mathcal{A}_{\mathcal{O},M}$ , one has that the sets  $A_K$  and the restrictions of the functions  $(f_j)_K$  to  $A_K$  are as in Theorem 3.3.2. Moreover, one can ensure that the restriction of  $F_K$  to any occurring  $A_K$  is as in Proposition 3.3.5.

Up to using a residual parametrization, the uniform version of Proposition 3.2.5 also holds, as follows.

**Proposition 4.5.4.** *Let  $X$  and  $U \subset X[m, 0, 0]$  be definable subassignments and let  $f_1, \dots, f_s$  be in  $\mathcal{C}^{\text{exp}}(U)$ . Then, up to replacing  $X$  by  $X_{\text{par}}$  for some well-chosen residual parametrization  $g$  of  $X$ ,  $U$  by  $U_{\text{par}} = g(U)$ , and the  $f_\ell$  correspondingly by  $(f_\ell)_{\text{par}}$ , the following holds. There exist an integer  $d \geq 0$ , definable morphisms  $h_{\ell i} : U \rightarrow h[1, 0, 0]$ , a definable surjection  $\varphi : U \rightarrow V \subset X[0, 0, t]$  for some  $t \geq 0$ , and functions  $G_{\ell i}$  in  $\mathcal{C}^{\text{exp}}(V)$  such that for each  $K$  in  $\mathcal{C}_{\mathcal{O}}$  with large enough residue field characteristic and for each  $\psi$  in  $\mathcal{D}_K$ , conditions 1) and 2) of Proposition 3.2.5 hold for  $d$ ,  $f_{\ell K, \psi}$ ,  $U_K$ ,  $V_K$ ,  $\varphi_K$ ,  $h_{\ell i K}$ , and  $G_{\ell i K, \psi}$ .*

*Proof.* Note that the uniform analogue of Lemma 3.3.6 holds, up to a well-chosen residual parameterization, by the same proof (with  $m = 1$  at the end), or, alternatively, by Theorem 2.2.2 of [9]. Now the proof of Proposition 3.2.5 works uniformly in  $K$  in  $\mathcal{A}_{\mathcal{O},M}$  for large enough  $M$ , where one uses Theorem 4.5.3 instead of Theorem 3.3.2 and Proposition 3.3.5. Note that the definition of  $\mathcal{C}^{\text{exp}}(\cdot)$ , and especially the part coming from  $\mathcal{Q}^{\text{exp}}(\cdot)$ , allows one, up to suitable residual parameterization, to apply the uniform analogue of Lemma 3.3.6 as in the  $p$ -adic case.  $\square$

From now on we will work and prove results for all  $K$  in  $\mathcal{C}_{\mathcal{O},M}$  instead of only in  $\mathcal{A}_{\mathcal{O},M}$ , which will be allowed by the uniform nature of the above results and by the classical Ax-Kochen principle of [1] for first order statements in the language  $\mathcal{L}_{\text{DP}}$ .

*Proof of Theorems 4.3.2 and 4.3.3.* If we allow ourselves to replace the given data using a residual parametrization  $g$  of  $X[m, 0, 0]$ , then the results follow from the respective results and techniques from Section 2.2 and the uniform  $p$ -adic Cell Decomposition Theorem 4.5.3, analogously to the  $p$ -adic case. Note that, in particular and up to a well-chosen residual parametrization, Proposition 3.4.3 and the techniques with skeletons go through in a uniform way. To get rid of the residual parametrization, one can proceed in various ways, depending of the objective. For Theorem 4.3.2 one uses squaring of the obtained constructible motivic functions  $h_i$  to get new ones with nonnegative values and one integrates over the residual variables that were introduced by the residual parametrization  $g$  to find the functions as desired (as in [8], Section 5.6, where this is called taking the push-forward along  $g^{-1}$ , and denoted by  $(g^{-1})_!$ ). For Theorems 4.3.3 and 4.3.1, one gets rid of the



residual parametrization by integrating over the residual variables that were introduced by the residual parametrization. This finishes the proofs.  $\square$

*Proof of Theorems 4.4.4 and 4.4.6.* If we allow ourselves an appropriate residual parametrization  $g$  to replace the original data, one gets the desired results as in the proof of Theorems 3.2.2, and 3.2.3, using the uniform analogues of the  $p$ -adic results and techniques, in particular, using Proposition 4.5.4 instead of Proposition 3.2.5. To remove the residual parametrization we proceed again differently for the two theorems. For Theorem 4.4.4 note that, for any motivic exponential function, say  $H$  in  $\mathcal{C}^{\exp}(Y)$ , there exists another motivic exponential function  $\overline{H}$  in  $\mathcal{C}^{\exp}(Y)$  such that for  $K \in \mathcal{C}_{\mathcal{O},M}$  (as usual for large enough  $M$ ) and for all  $y \in Y_K$ , one has that  $H_K(y)$  is the complex conjugate of  $\overline{H}_K(y)$ , where  $\overline{H}$  can be obtained from  $H$  by changing the signs in the arguments of the additive characters. Hence, by replacing  $H$  by  $H \cdot \overline{H}$ , one can assume that, as far as the zero locus of  $H$  is concerned, that  $H$  takes real, nonnegative values. Now just integrate motivically (in the sense of taking the push-forward as in Section 3.6 of [9] along  $g^{-1}$ ) over those residue field variables that were created by the above residual parametrization  $g$  to obtain the desired results. This finishes the argument for Theorem 4.4.4. For Theorem 4.4.6, one just integrates over the residue field variables that were created by the residual parametrization.  $\square$

*Proof of Theorems 4.3.1 and 4.4.3.* As the proofs of Theorems 3.1.1 and 3.2.1, using Theorems 4.3.3 and 4.4.6 instead of Theorems 3.1.4 and 3.2.3, together with Theorem 9.1.4 of [9].  $\square$

*Proof of Theorems 4.4.1 and 4.4.2.* For the first statement of Theorem 4.4.1, resp. of Theorem 4.4.2, take  $h_1$ , resp.  $h_2$ , as given by Theorem 4.4.4. For the second statement of Theorem 4.4.1, resp. of Theorem 4.4.2, take  $h_1$ , resp.  $h_2$ , as given by Corollary 3.2.4. In all cases the proof is finished by applying the transfer principle Proposition 9.2.1 of [9] to  $h_1$  and  $h_2$ .  $\square$

Let us finally indicate how to obtain the Transfer Principles that are quoted in the introduction (above Theorem 1.0.1), where we considered all non-trivial additive characters on  $K$  while in Theorem 1.0.1, only characters from  $\mathcal{D}_K$  are considered. These quoted transfer principles easily follow by noting that all non-trivial additive characters can be given in a family which falls under the scope of Theorem 4.4.1.

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## REFERENCES

1. J. Ax and S. Kochen, *Diophantine problems over local fields. II.: A complete set of axioms for  $p$ -adic number theory*, Amer. J. Math. **87** (1965), 631–648.
2. R. Cluckers, *Presburger sets and  $p$ -minimal fields*, J. of Symbolic Logic **68** (2003), 153–162, arXiv:math.LO/0206197.
3. ———, *Analytic  $p$ -adic cell decomposition and integrals*, Trans. Amer. Math. Soc. **356** (2004), no. 4, 1489 – 1499, arXiv:math.NT/0206161.
4. R. Cluckers, G. Comte, and F. Loeser, *Lipschitz continuity properties for  $p$ -adic semi-algebraic and subanalytic functions*, GAFA (Geom. Funct. Anal.) **20** (2010), 68–87, arXiv:0904.3853.
5. R. Cluckers, J. Gordon, and I. Halupczok, *Definability results for invariant distributions on a reductive unramified  $p$ -adic group*, preprint (2011).
6. R. Cluckers, T. Hales, and F. Loeser, *Bookchapter in: Stabilisation de la formule des traces, variétés de Shimura, et applications arithmétiques, I.*, ch. Transfer Principle for the Fundamental Lemma, International Press of Boston, 2011.
7. R. Cluckers and L. Lipshitz, *Fields with analytic structure*, J. Eur. Math. Soc. (JEMS) **13** (2011), 1147–1223, math.LO/0610666.
8. R. Cluckers and F. Loeser, *Constructible motivic functions and motivic integration*, Inventiones Mathematicae **173** (2008), no. 1, 23–121, arxiv:math.AG/0410203.
9. ———, *Constructible exponential functions, motivic Fourier transform and transfer principle*, Annals of Mathematics **171** (2010), no. 2, 1011–1065, arXiv:math.AG/0512022.
10. R. Cluckers and D. J. Miller, *Diagrams of Lebesgue classes of real constructible functions*, (2011), preprint.
11. ———, *Loci of integrability, zero loci, and stability under integration for constructible functions on euclidean space with lebesgue measure*, Int. Math. Res. Not. IMRN (2011), doi: 10.1093/imrn/rnr133.
12. J. Denef, *On the evaluation of certain  $p$ -adic integrals*, Théorie des nombres, Sémin. Delange-Pisot-Poitou 1983–84, vol. 59, 1985, pp. 25–47.
13. ———,  *$p$ -adic semialgebraic sets and cell decomposition*, Journal für die reine und angewandte Mathematik **369** (1986), 154–166.
14. J. Denef and L. van den Dries,  *$p$ -adic and real subanalytic sets*, Annals of Mathematics **128** (1988), no. 1, 79–138.
15. L. van den Dries, *Tame topology and o-minimal structures*, Lecture note series, vol. 248, Cambridge University Press, 1998.
16. L. van den Dries, D. Haskell, and D. Macpherson, *One-dimensional  $p$ -adic subanalytic sets*, Journal of the London Mathematical Society **59** (1999), no. 1, 1–20.
17. H. Glöckner, *Comparison of some notions of  $C^k$ -maps in multi-variable non-archimedean analysis*, Bull. Belg. Math. Soc. Simon Stevin **14** (2007), no. 5, 877–904.
18. Harish-Chandra, *Admissible invariant distributions on reductive  $p$ -adic groups*, University Lecture Series, vol. 16, American Mathematical Society, Providence, RI, 1999, Preface and notes by Stephen DeBacker and Paul J. Sally, Jr.
19. E. Hrushovski and D. Kazhdan, *Integration in valued fields*, Algebraic geometry and number theory, Progr. Math., vol. 253, Birkhäuser Boston, Boston, MA, 2006, pp. 261–405.
20. B. Lemaire, *Intégrabilité locale des caractères-distributions de  $GL_n(f)$  où  $f$  est un corps local non-archimédien de caractéristique quelconque*, Compositio Math. **100** (1996), no. 1, 41–75.
21. A. Macintyre, *On definable subsets of  $p$ -adic fields*, Journal of Symbolic Logic **41** (1976), 605–610.
22. Bao Châu Ngô, *Le lemme fondamental pour les algèbres de Lie*, Publ. Math. Inst. Hautes Études Sci. (2010), no. 111, 1–169. MR **2653248** (2011h:22011)

23. J. Pas, *Uniform  $p$ -adic cell decomposition and local zeta functions*, Journal für die reine und angewandte Mathematik **399** (1989), 137–172.
24. M. Presburger, *On the completeness of a certain system of arithmetic of whole numbers in which addition occurs as the only operation*, History and Philosophy of Logic **12** (1991), no. 2, 92–101.
25. F. Rodier, *Intégrabilité locale des caractères du groupe  $\mathrm{gl}(n, k)$  où  $k$  est un corps local de caractéristique positive*, Duke Math. J. **52**, no. 3, 771–792 (French).
26. J.-L. Waldspurger, *Endoscopie et changement de caractéristique*, J. Inst. Math. Jussieu **5** (2006), no. 3, 423–525.
27. A. J. Wilkie, *Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function*, J. Amer. Math. Soc. **9** (1996), no. 4, 1051–1094. MR **1398816** (98j:03052)

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